Implementation of Discrete-Time Systems

The two important forms of expressing system leading to different realizations of FIR & IIR filters are

a) Difference equation form

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=1}^{M} b_k\, x(n-k)$$

b) Ration of polynomials

$$H(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1 + \sum_{k=1}^{N} a_k Z^{-k}}$$

The different factors that influence choice of a specific realization are

- Computational complexity
- Memory requirements
- Finite-word-length
- Pipeline / parallel processing

Computation Complexity

- Numbers of arithmetic operations i.e multiplication, addition & divisions
- In the recent processors the fetch time from memory & number of times a comparison between two numbers is performed per output sample is also considered and found to be important

Memory requirements

- This is basically number of memory locations required to store the system parameters, past inputs, past outputs, and any intermediate computed values.

Finite-word-length effects

- These effects refer to the quantization effects that are inherent in any digital implementation of the system, either in hardware or in software.
- Basically effect of truncation & rounding-off of samples
- The extent of this effect varies with type of arithmetic used(fixed or floating)
- The effects have influence on system characteristics.
- A structure which is less sensitive to this effect need to be chosen.

Pipeline / Parallel Processing

- Suitability of the structure for pipelining & parallel processing is considered.

Structure for FIR Systems

FIR system is described by,

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

Or equivalently, the system function

$$H(Z) = \sum_{k=0}^{M-1} b_k Z^{-k}$$

Where we can identify $h(n) = \begin{cases} b_n & 0 \le n \le n-1 \\ 0 & otherwise \end{cases}$

Different FIR Structures

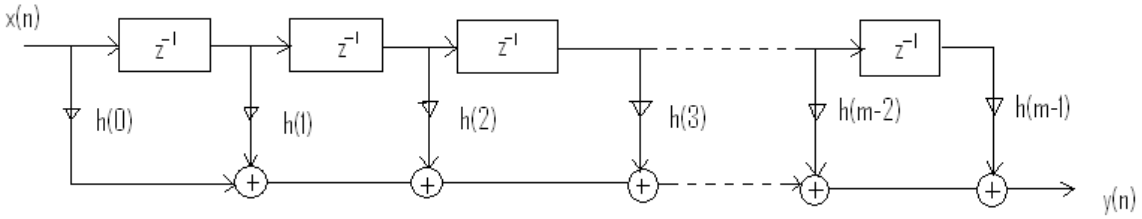
1. Direct form
2. Cascade form
3. Frequency-sampling realization
4. Lattice realization

Direct – Form Structure

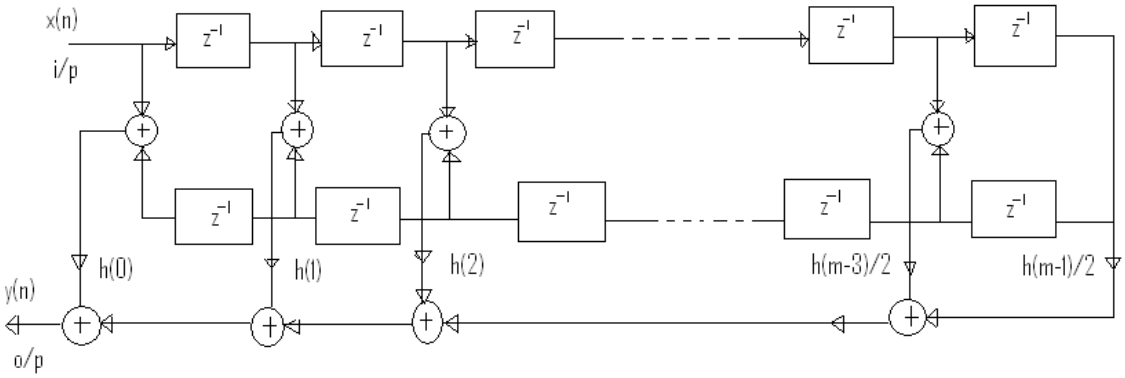
Convolution formula is used to express FIR system given by,

$$y(n) = \sum_{k=0}^{M-1} h(k)\ x(n-k)$$

- Non recursive structure



- Requires M-1 memory locations for storing the M-1 previous inputs
- Computationally need M multiplications and M-1 additions per output point
- Referred to as tapped delay line or transversal system
- Efficient structure for linear phase FIR filters are possible where $h(n) = \pm h(M-1-n)$



PROBLEM

Realize the following system function using minimum number of multiplication

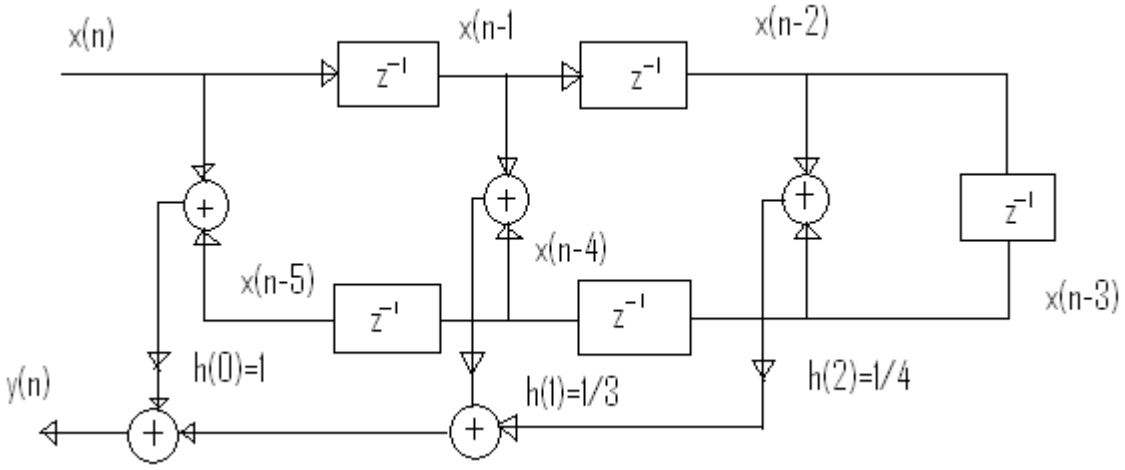
(1) $H(Z) = 1 + \frac{1}{3}Z^{-1} + \frac{1}{4}Z^{-2} + \frac{1}{4}Z^{-3} + \frac{1}{3}Z^{-4} + Z^{-5}$

We recognize $h(n) = \left[1, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3}, 1\right]$

M is even = 6, and we observe h(n) = h(M-1-n) h(n) = h(5-n)

i.e h(0) = h(5) h(1) = h(4) h(2) = h(3)

Direct form structure for Linear phase FIR can be realized



Exercise: Realize the following using system function using minimum number of multiplication.

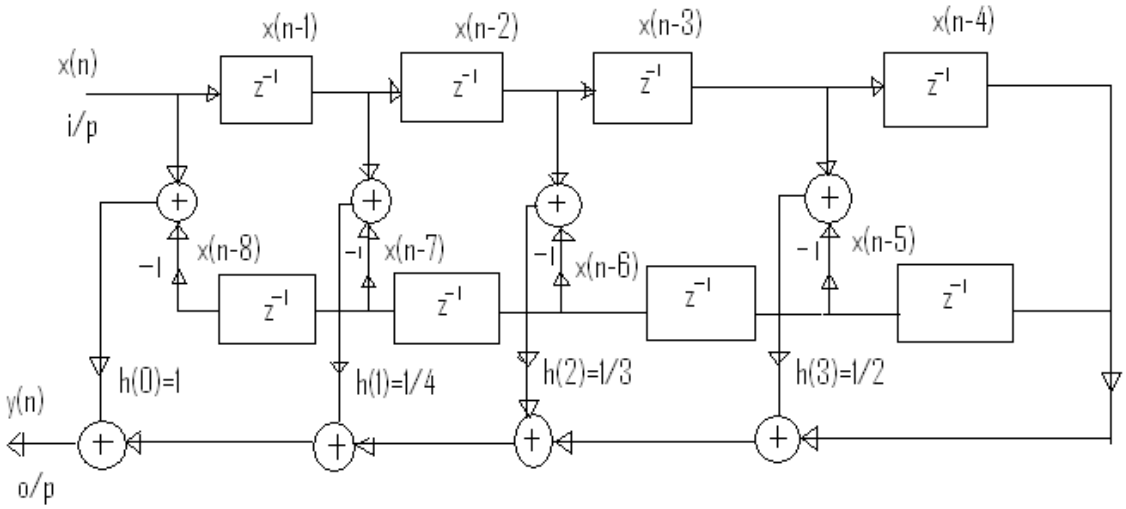
$$H(Z) = 1 + \frac{1}{4}Z^{-1} + \frac{1}{3}Z^{-2} + \frac{1}{2}Z^{-3} - \frac{1}{2}Z^{-5} - \frac{1}{3}Z^{-6} - \frac{1}{4}Z^{-7} - Z^{-8}$$

m=9 $h(n) = \left[1, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -1\right]$

odd symmetry

h(n) = -h(M-1-n); h(n) = -h(8-n); h(m-1/2) = h(4) = 0

h(0) = -h(8); h(1) = -h(7); h(2) = -h(6); h(3) = -h(5)



Cascade – Form Structure

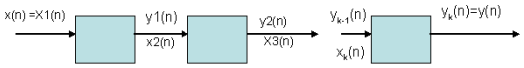
The system function H(Z) is factored into product of second – order FIR system

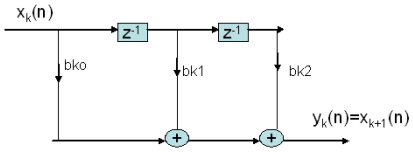
$$H(Z) = \prod_{k=1}^{K} H_k(Z)$$

Where $H_k(Z) = b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2}$ k = 1, 2, ….. K

and K = integer part of (M+1) / 2

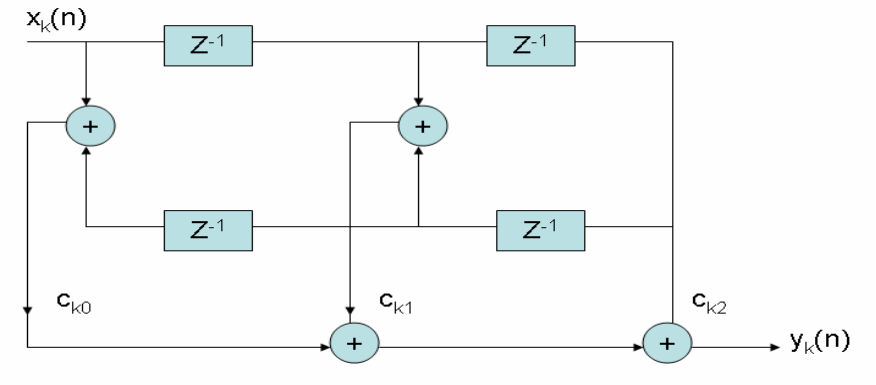
- The filter parameter b_0 may be equally distributed among the K filter section, such that $b_0 = b_{10}\, b_{20} \ldots b_{k0}$ or it may be assigned to a single filter section
- The zeros of H(z) are grouped in pairs to produce the second – order FIR system
- Form pairs of complex-conjugate roots so that the coefficients $\{b_{ki}\}$ are real valued.





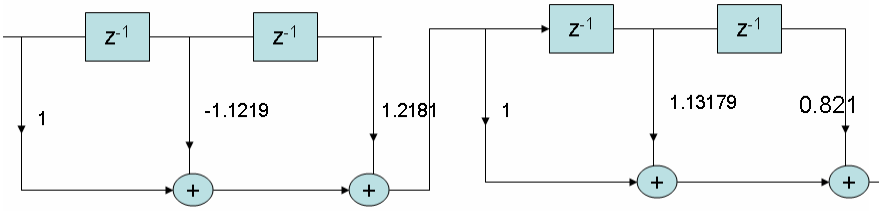
- In case of linear –phase FIR filter, the symmetry in h(n) implies that the zeros of H(z) also exhibit a form of symmetry
- If zk and zk* are pair of complex – conjugate zeros then 1/zk and 1/zk* are also a pair complex –conjugate zeros. Thus simplified fourth order sections are formed.

- $$H_k(z) = C_{k0}(1 - z_k z^{-1})(1 - z_k * z^{-1})(1 - z^{-1}/z_k)(1 - z^{-1}/z_{k*})$$
$$= C_{k0} + C_{k1}z^{-1} + C_{k2}z^{-2} + C_{k1}z^{-3} + z^{-4}$$



Problem: Realize the difference equation
$y(n) = x(n) + 0.25x(n-1) + 0.5x(n-2) + 0.75x(n-3) + x(n-4)$
in cascade form.

Soln:
$$Y(z) = X(z)\{1 + 0.25z^{-1} + 0.5z^{-2} + 0.75z^{-3} + z^{-4})$$
$$H(z) = 1 + 0.25z^{-1} + 0.5z^{-2} + 0.75z^{-3} + z^{-4}$$
$$H(z) = (1 - 1.1219z^{-1} + 1.2181z^{-2})(1 + 1.3719z^{-1} + 0.821z^{-2})$$
$$H(z) = H_1(z)H_2(z)$$



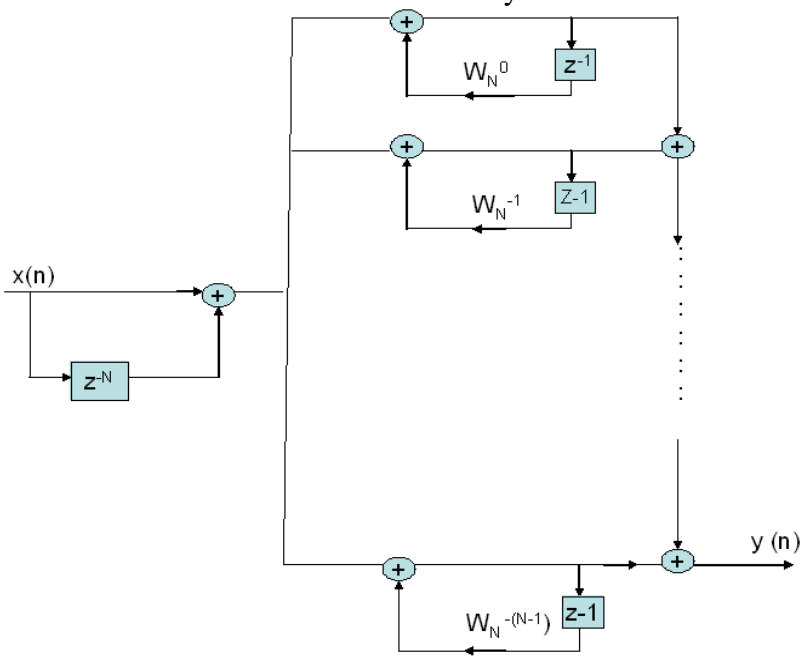
Frequency Sampling realization:

We can express system function H(z) in terms of DFT samples H(k) which is given by

$$H(z) = (1 - z^{-N})\frac{1}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1 - W_N^{-k}z^{-1}}$$

This form can be realized with cascade of FIR and IIR structures. The term $(1-z^{-N})$ is realized as FIR and the term $\frac{1}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1-W_N^{-k}z^{-1}}$ as IIR structure.

The realization of the above freq sampling form shows necessity of complex arithmetic. Incorporating symmetry in h(n) and symmetry properties of DFT of real sequences the realization can be modified to have only real coefficients.



Lattice structures

Lattice structures offer many interesting features:

1. Upgrading filter orders is simple. Only additional stages need to be added instead of redesigning the whole filter and recalculating the filter coefficients.
2. These filters are computationally very efficient than other filter structures in a filter bank applications (eg. Wavelet Transform)
3. Lattice filters are less sensitive to finite word length effects.

Consider

$$H(z)=\frac{Y(z)}{X(z)}=1+\sum_{i=1}^{m}a_m(i)z^{-i}$$

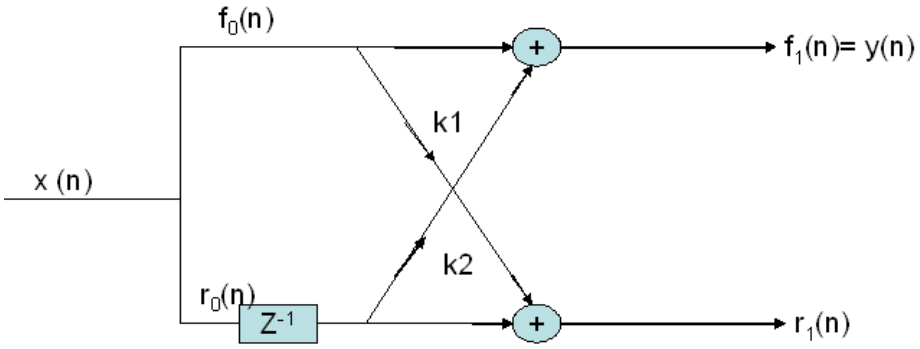
m is the order of the FIR filter and $a_m(0)=1$

when m = 1 $Y(z)/X(z) = 1+ a_1(1)\, z^{-1}$

$y(n)= x(n)+ a_1(1)x(n-1)$

f1(n) is known as upper channel output and r1(n)as lower channel output.

$f_0(n)= r_0(n)=x(n)$



The outputs are

$$f_1(n) = f_0(n) + k_1 r_0(n-1) \qquad 1a$$
$$r_1(n) = k_1 f_0(n) + r_0(n-1) \qquad 1b$$
$$if \quad k_1 = a_1(1), then \quad f_1(n) = y(n)$$

If m=2

$$\frac{Y(z)}{X(z)} = 1 + a_2(1)z^{-1} + a_2(2)z^{-2}$$
$$y(n) = x(n) + a_2(1)x(n-1) + a_2(2)x(n-2)$$
$$y(n) = f_1(n) + k_2 r_1(n-1) \qquad (2)$$

Substituting 1a and 1b in (2)

$$\begin{aligned} y(n) &= f_0(n) + k_1 r_0(n-1) + k_2[k_1 f_0(n-1) + r_0(n-2)] \\ &= f_0(n) + k_1 r_0(n-1) + k_2 k_1 f_0(n-1) + k_2 r_0(n-2)] \end{aligned}$$
$$\text{sin } ce \quad f_0(n) = r_0(n) = x(n)$$
$$\begin{aligned} y(n) &= x(n) + k_1 x(n-1) + k_2 k_1 x(n-1) + k_2 x(n-2)] \\ &= x(n) + (k_1 + k_1 k_2)x(n-1) + k_2 x(n-2) \end{aligned}$$

We recognize

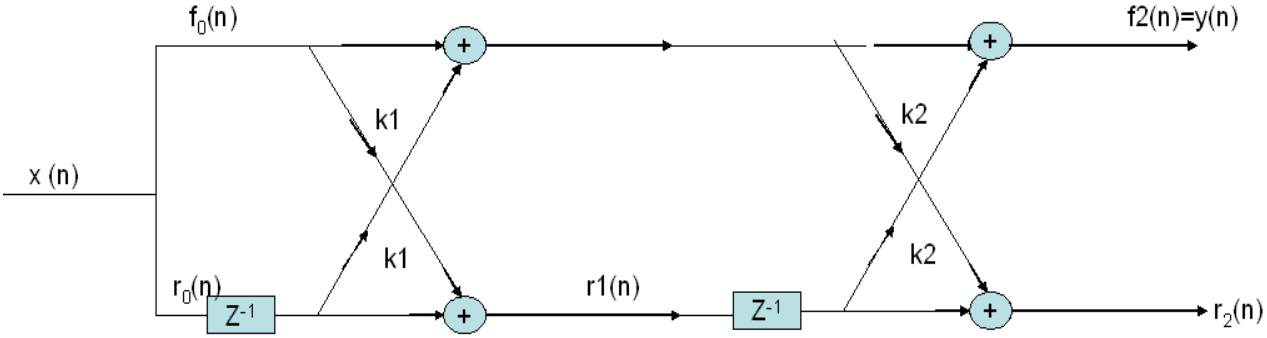
$a_2(1) = k_1 + k_1 k_2$

$a_2(1) = k_2$

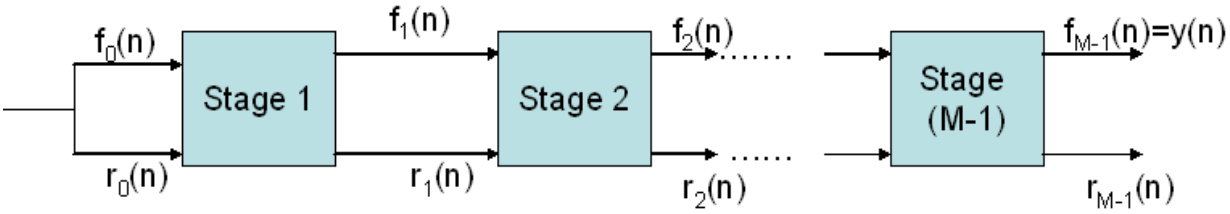
Solving the above equation we get

$$k_1 = \frac{a_2(1)}{1 + a_2(2)} \quad and \quad k_2 = a_2(2) \qquad (4)$$

Equation (3) means that, the lattice structure for a second-order filter is simply a cascade of two first-order filters with k1 and k2 as defined in eq (4)



Similar to above, an Mth order FIR filter can be implemented by lattice structures with M – stages



Direct Form –I to lattice structure

For m = M, M-1, ……..2, 1 do

$$k_m = a_m(m)$$

$$a_{m-1}(i) = \frac{a_m(i) - a_m(m)a_m(m-i)}{1-k_m^2} \qquad 1 \le i \le m-1$$

- The above expression fails if $k_m=1$. This is an indication that there isa zero on the unit circle. If $k_m=1$, factor out this root from A(z) and the recursive formula can be applied for reduced order system.

$$for\ m = 2\ and\ m = 1$$

$$k_2 = a_2(2) \quad \& \quad k_1 = a_1(1)$$

$$for\ m = 2\ \&\ i = 1$$

$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1-k_2^2} = \frac{a_2(1)[1-a_2(2)]}{1-a_2^2(2)} = \frac{a_2(1)}{1+a_2(2)}$$

$$Thus \quad k_1 = \frac{a_2(1)}{1+a_2(2)}$$

Lattice to direct form –I

For m = 1,2,…….M-1

$$a_m(0) = 1$$

$$a_m(m) = k_m$$

$$a_m(i) = a_{m-1}(i) + a_m(m)a_{m-1}(m-i) \quad 1 \le i \le m-1$$

Problem:

Given FIR filter $H(Z) = 1 + 2Z^{-1} + \frac{1}{3}Z^{-2}$ obtain lattice structure for the same

Given $a_1(1) = 2$, $a_2(2) = \frac{1}{3}$

Using the recursive equation for

m = M, M-1, ……, 2, 1

here M=2 therefore m = 2, 1

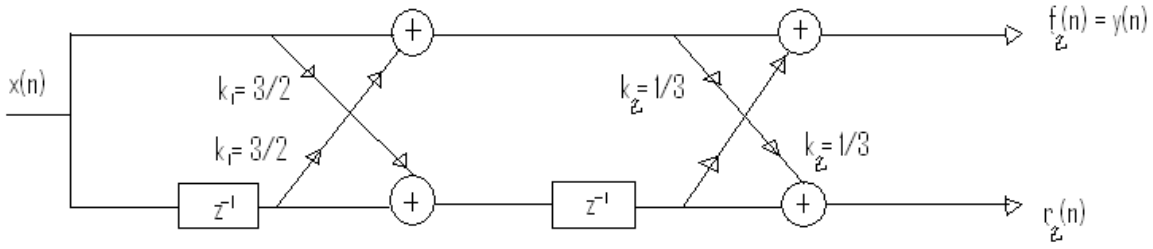
if m=2 $k_2 = a_2(2) = \frac{1}{3}$

if m=1 $k_1 = a_1(1)$

also, when m=2 and i=1

$$a_1(1) = \frac{a_2(1)}{1+a_2(2)} = \frac{2}{1+\frac{1}{3}} = \frac{3}{2}$$

Hence $k_1 = a_1(1) = \frac{3}{2}$



Problem:

Consider an FIR lattice filter with co-efficients $k_1 = \frac{1}{2}$, $k_2 = \frac{1}{3}$, $k_3 = \frac{1}{4}$. Determine the FIR filter co-efficient for the direct form structure
$(H(Z) = a_3(0) + a_3(1)Z^{-1} + a_3(2)Z^{-2} + a_3(3)Z^{-3})$

$a_3(0) = 1$ $\quad a_3(3) = k_3 = 1/4$ $\quad a_2(2) = k_2 = \frac{1}{3}$

$a_1(1) = k_1 = \frac{1}{2}$

for m=2, i=1

$$a_2(1) = a_1(1) + a_2(2)a_1(1)$$

$$= a_1(1)[1 + a_2(2)] = \frac{1}{2}\left[1 + \frac{1}{3}\right]$$

$$= \frac{4}{6} = \frac{2}{3}$$

for m=3, i=1

$$a_3(1) = a_2(1) + a_3(3)a_2(2)$$

$$= \frac{2}{3} + \frac{1}{4}.\frac{1}{3}$$

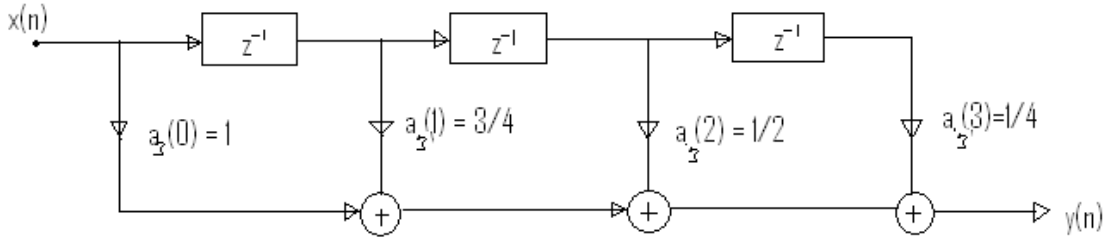
$$= \frac{2}{3} + \frac{1}{12} = \frac{8+1}{12}$$

$$= \frac{9}{12} = \frac{3}{4}$$

for m=3 & i=2

$$a_3(2) = a_2(2) + a_3(3)a_2(1)$$
$$= \frac{1}{3} + \frac{1}{4}.\frac{2}{3}$$
$$= \frac{1}{3} + \frac{1}{6} = \frac{2+1}{6}$$
$$= \frac{3}{6} = \frac{1}{2}$$

$a_3(0) = 1$, $a_3(1) = \frac{3}{4}$, $a_3(2) = \frac{1}{2}$, $a_3(3) = \frac{1}{4}$



Structures for IIR Filters

The IIR filters are represented by system function;

$$H(Z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

and corresponding difference equation given by,

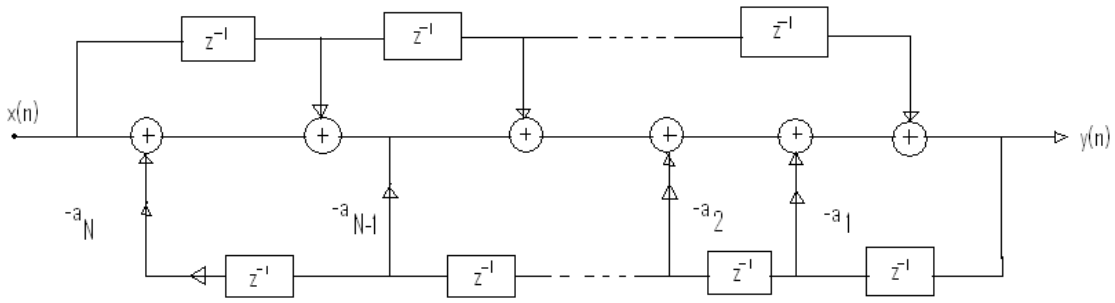
$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{N} b_k x(n-k)$$

Different realization

1. Direct form-I
2. Direct form-II
3. Cascade form
4. Parallel form
5. Lattice form

Direct form-I

- Straight forward Implementation of difference equation
- Simple



Direct form-II

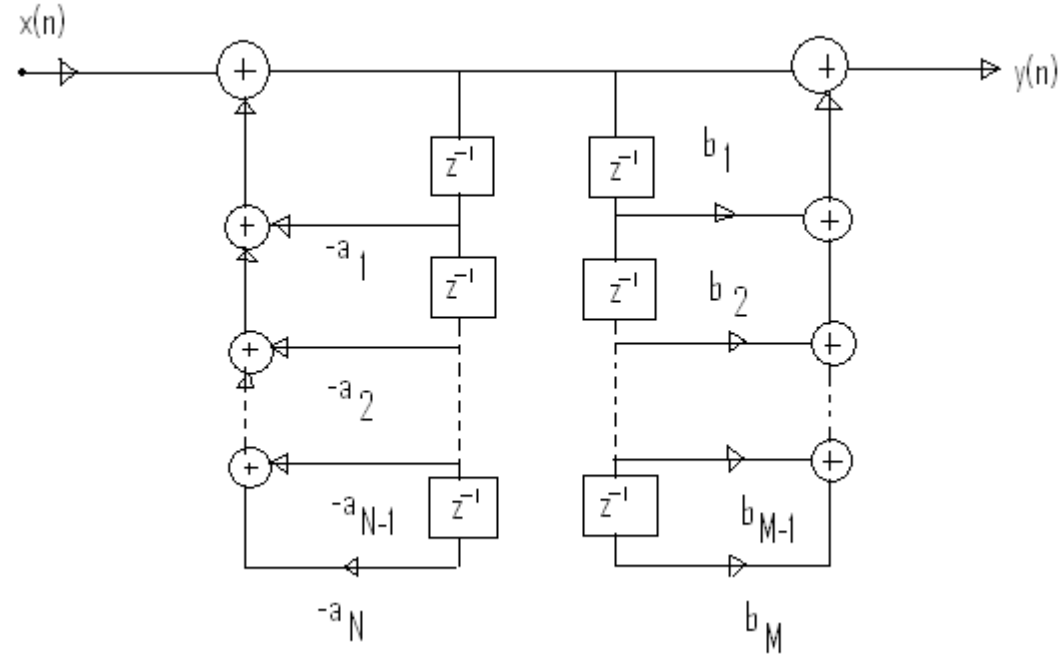
$$H(z) = \frac{Y(z)}{X(z)} = \frac{V(z)}{X(z)} \cdot \frac{Y(z)}{V(z)}$$

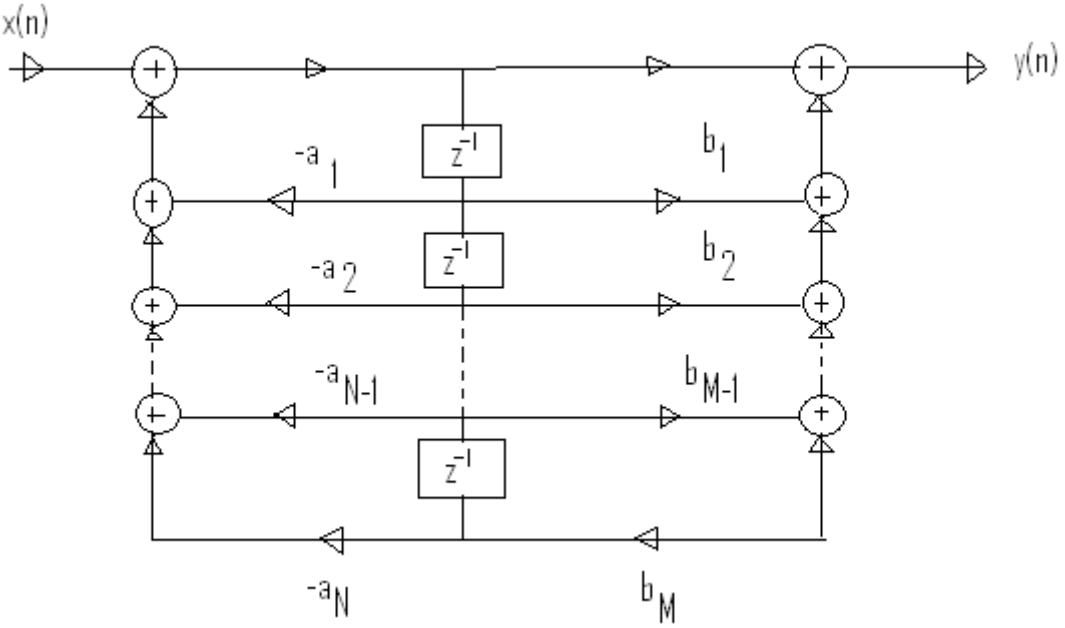
$$\frac{V(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$ ----------all poles

$$\frac{Y(z)}{V(z)} = \left(1 + \sum_{k=1}^{M} b_k z^{-k}\right)$$ ----------all zeros

$$v(n) = x(n) - \sum_{k=1}^{N} a_k v(n-k)$$

$$y(n) = v(n) + \sum_{k=1}^{M} b_k v(n-1)$$





- This realization requires M+N+! multiplication, M+N addition and the maximum of {M, N} memory location

Cascade Form

$H(z) = H_1(z)H_2(z)....H_k(z)$

Where $H_k(Z)$ could be first order or second order section realized in Direct form – II form i.e.,

If $H_k(Z) = \dfrac{b_{k0} + b_{k1}Z^{-1} + b_{k2}Z^{-2}}{1 + a_{k1}Z^{-1} + a_{k2}Z^{-2}}$

And K is the integer part of (N+1)/2

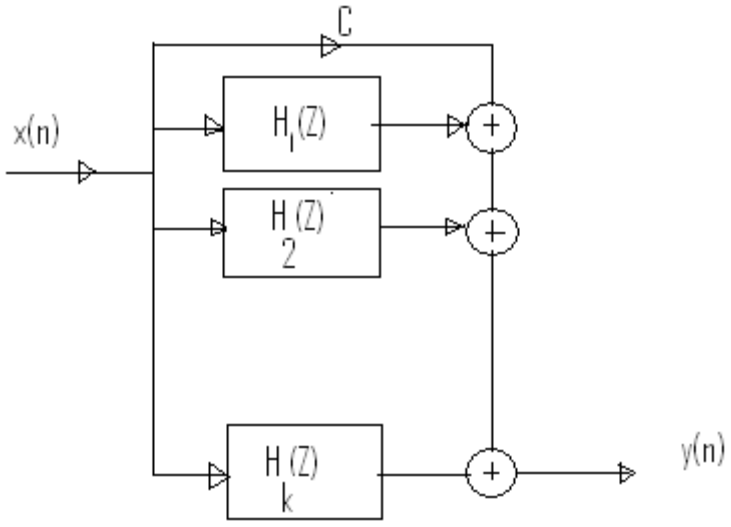
- Similar to FIR cascade realization, the parameter b_0 can be distributed equally among the k filter section B_0 that $b_0 = b_{10}b_{20}\ldots..b_{k0}$
- Second order sections are required to realize section which has complex-conjugate poles with real co-efficients.
- Pairing the two complex-conjugate poles with a pair of complex-conjugate zeros or real-valued zeros to form a subsystem of the type shown above done arbitrarily.
- Although all cascade realizations are equivalent for infinite precision arithmetic, the various realizations may differ significantly when implemented with finite precision arithmetic.

Parallel form structure

If $N \geq M$ we can express system function

$$H(Z) = C + \sum_{k=1}^{N} \frac{A_k}{1 - p_k Z^{-1}} \qquad = C + \sum_{k=1}^{N} H_k(Z)$$

Where $\{p_k\}$ are the poles, $\{A_k\}$ are the coefficients in the partial fraction expansion, and the constant C is defined as $C = b_N / a_N$



Where $H_k(Z) = \dfrac{b_{k0} + b_{k1}Z^{-1}}{1 + a_{k1}Z^{-1} + a_{k2}Z^{-2}}$

Problem :

Determine the

(i)Direct form-I (ii) Direct form-II (iii) Cascade &

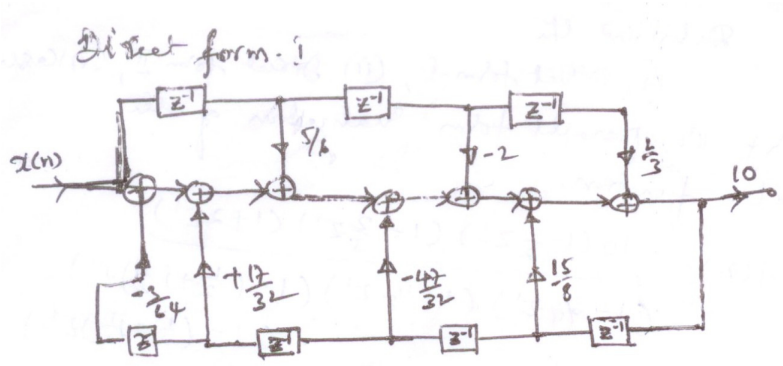
(iv)Parallel form realization of the system function

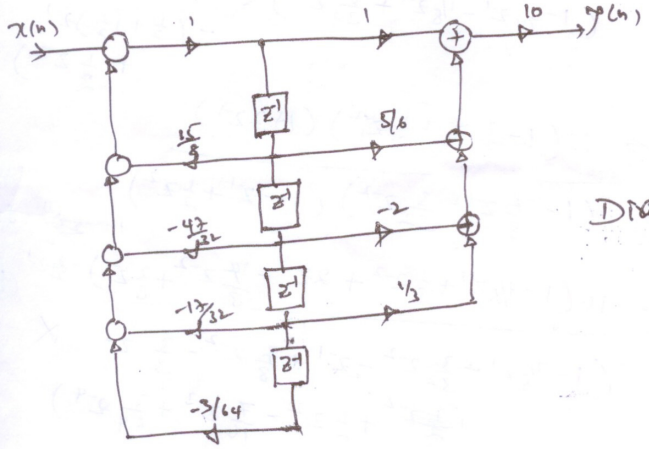
$$H(Z)=\frac{10\left(1-\frac{1}{2}Z^{-1}\right)\left(1-\frac{2}{3}Z^{-1}\right)\left(1+2Z^{-1}\right)}{\left(1-\frac{3}{4}Z^{-1}\right)\left(1-\frac{1}{8}Z^{-1}\right)\left(1-\left(\frac{1}{2}+j\frac{1}{2}\right)Z^{-1}\right)\left(1-\left(\frac{1}{2}-j\frac{1}{2}\right)Z^{-1}\right)}$$

$$=\frac{10\left(1-\frac{7}{6}Z^{-1}+\frac{1}{3}Z^{-2}\right)\left(1+2Z^{-1}\right)}{\left(1+\frac{7}{8}Z^{-1}+\frac{3}{32}Z^{-2}\right)\left(1-Z^{-1}+\frac{1}{2}Z^{-2}\right)}$$

$$H(Z)=\frac{10\left(1+\frac{5}{6}Z^{-1}-2Z^{-2}+\frac{2}{3}Z^{-3}\right)}{\left(1-\frac{15}{8}Z^{-1}+\frac{47}{32}Z^{-2}-\frac{17}{32}Z^{-3}+\frac{3}{64}Z^{-4}\right)}$$

$$H(z)=\frac{(-14.75-12.90z^{-1})}{(1+\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2})}+\frac{(24.50+26.82z^{-1})}{(1-z^{-1}+\frac{1}{2}z^{-2})}$$



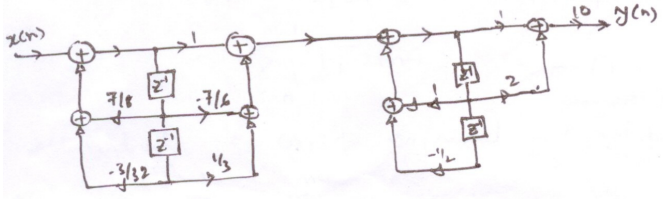


Direct form-II

Cascade form

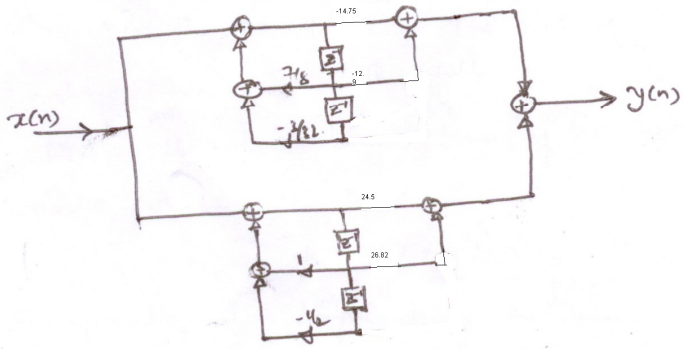
$$H(z) = H_1(z)\ H_2(z)$$

$$H_1(z) = \frac{1 - \frac{7}{6}z^{-1} + \frac{1}{3}z^{-2}}{\left(1 - \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2}\right)} \qquad H_2(z) = \frac{10(1+2z^{-1})}{(1 - z^{-1} + \frac{1}{2}z^{-2})}$$



Parallel form

$$H(z) = H_1(z) + H_2(z)$$

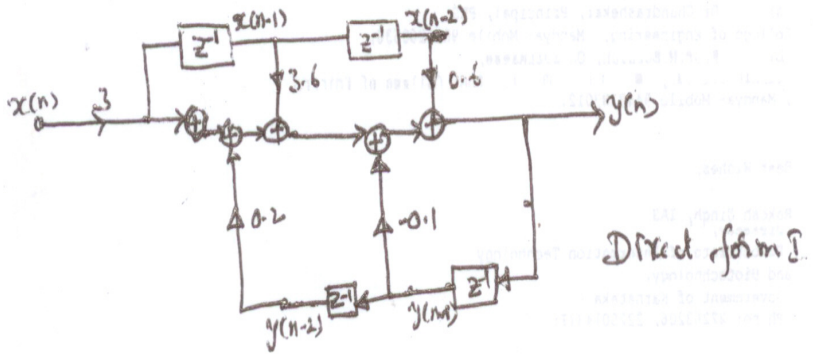


Problem:

Obtain the direct form – I, direct form-II

Cascade and parallel form realization for the following system,

$y(n)= -0.1\ y(n-1)+0.2y(n-2)+3x(n)+3.6\ x(n-1)+0.6\ x(n-2)$

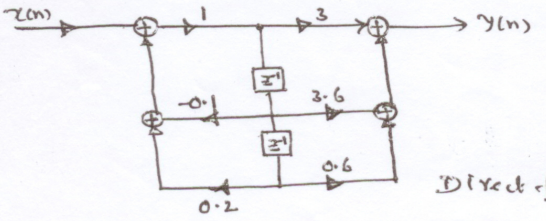


Direct form I

Direct form II

Taking ZT on both sides and finding H(z)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

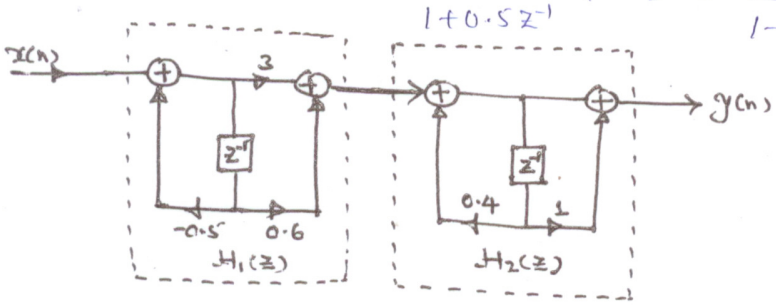


Direct form II

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$= \frac{(3 + 0.6z^{-1})(1 + z^{-1})}{(1 + 0.5z^{-1})(1 - 0.4z^{-1})}$$

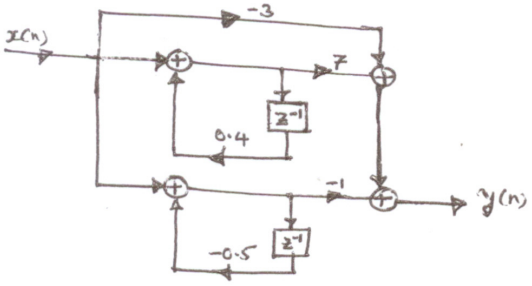
Let $H_1(z) = \frac{3 + 0.6z^{-1}}{1 + 0.5z^{-1}}$ & $H_2(z) = \frac{1 + z^{-1}}{1 - 0.4z^{-1}}$



Parallel form

$$H(z) = -3 + \frac{7}{1 - 0.4z^{-1}} - \frac{1}{1 + 0.5z^{-1}}$$

$$= C + H_1(z) + H_2(z)$$



Lattice Structure for IIR System

Consider an All-pole system with system function.

$$H(Z) = \frac{1}{1 + \sum_{k=1}^{N} a_N(k) Z^{-k}} = \frac{1}{A_N(Z)}$$

The corresponding difference equation for this IIR system is,

$$y(n) = -\sum_{k=1}^{N} a_N(k)\, y(n-k) + x(n)$$

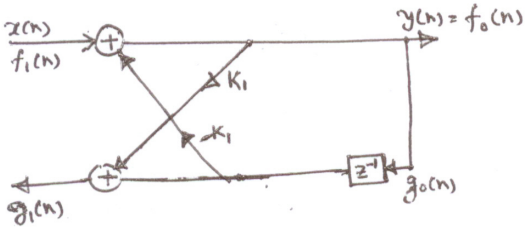
OR

$$x(n) = y(n) + \sum_{k=1}^{N} a_N(k)\, y(n-k)$$

For N=1

$$x(n) = y(n) + a_1(1)\, y(n-1)$$

Which can realized as,



We observe

$$x(n) = f_1(n)$$

$$y(n) = f_0(n) = f_1(n) - k_1 g_0(n-1)$$

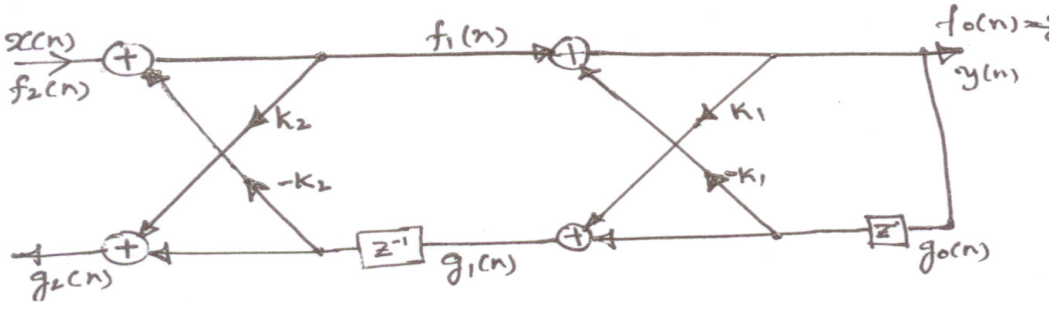
$$= x(n) - k_1 y(n-1) \qquad k_1 = a_1(1)$$

$$g_1(n) = k_1 f_0(n) + g_0(n-1) = k_1 y(n) + y(n-1)$$

For N=2, then

$$y(n) = x(n) - a_2(1)\, y(n-1) - a_2(2)\, y(n-2)$$

This output can be obtained from a two-stage lattice filter as shown in below fig



$$f_2(n) = x(n)$$

$$f_1(n) = f_2(n) - k_2 g_1(n-1)$$

$$g_2(n) = k_2 f_1(n) + g_1(n-1)$$

$$f_0(n) = f_1(n) - k_1 g_0(n-1)$$

$$g_1(n) = k_1 f_0(n) + g_0(n-1)$$

$$y(n) = f_0(n) = g_0(n) = f_1(n) - k_1 g_0(n-1)$$

$$= f_2(n) - k_2 g_1(n-1) - k_1 g_0(n-1)$$

$$= f_2(n) - k_2\left[k_1 f_0(n-1) + g_0(n-2)\right] - k_1 g_0(n-1)$$
$$= x(n) - k_2\left[k_1 y(n-1) + y(n-2)\right] - k_1 y(n-1)$$
$$= x(n) - k_1(1+k_2)y(n-1) - k_2 y(n-2)$$

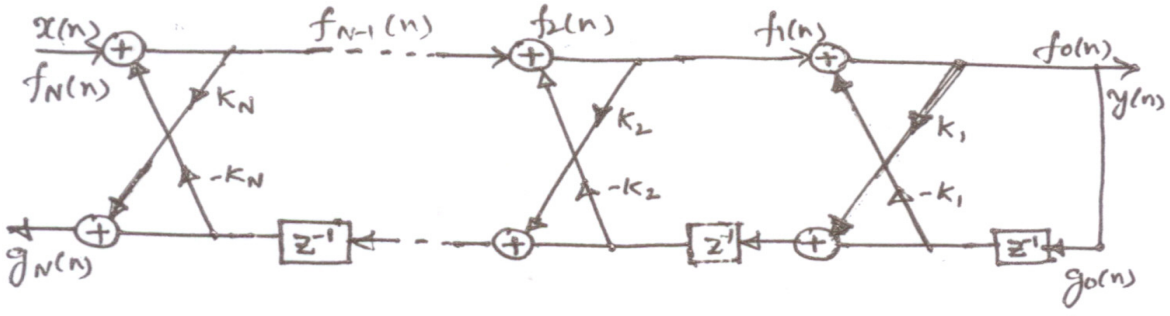
Similarly

$$g_2(n) = k_2 y(n) + k_1(1+k_2)y(n-1) + y(n-2)$$

We observe

$$a_2(0) = 1; a_2(1) = k_1(1+k_2); a_2(2) = k_2$$

N-stage IIR filter realized in lattice structure is,



$$f_N(n) = x(n)$$
$$f_{m-1}(n) = f_m(n) - k_m g_{m-1}(n-1) \qquad \text{m=N, N-1,---1}$$
$$g_m(n) = k_m f_{m-1}(n) + g_{m-1}(n-1) \qquad \text{m=N, N-1,---1}$$
$$y(n) = f_0(n) = g_0(n)$$

Conversion from lattice structure to direct form

$$a_m(m) = k_m; \qquad a_m(0) = 1$$
$$a_m(k) = a_{m-1}(k) + a_m(m)a_{m-1}(m-k)$$

Conversion from direct form to lattice structure

$$a_{m-1}(0) = 1 \qquad k_m = a_m(m)$$

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)\, a_m(m-k)}{1 - a_m^2(m)}$$

Lattice – Ladder Structure

A general IIR filter containing both poles and zeros can be realized using an all pole lattice as the basic building block.

If,

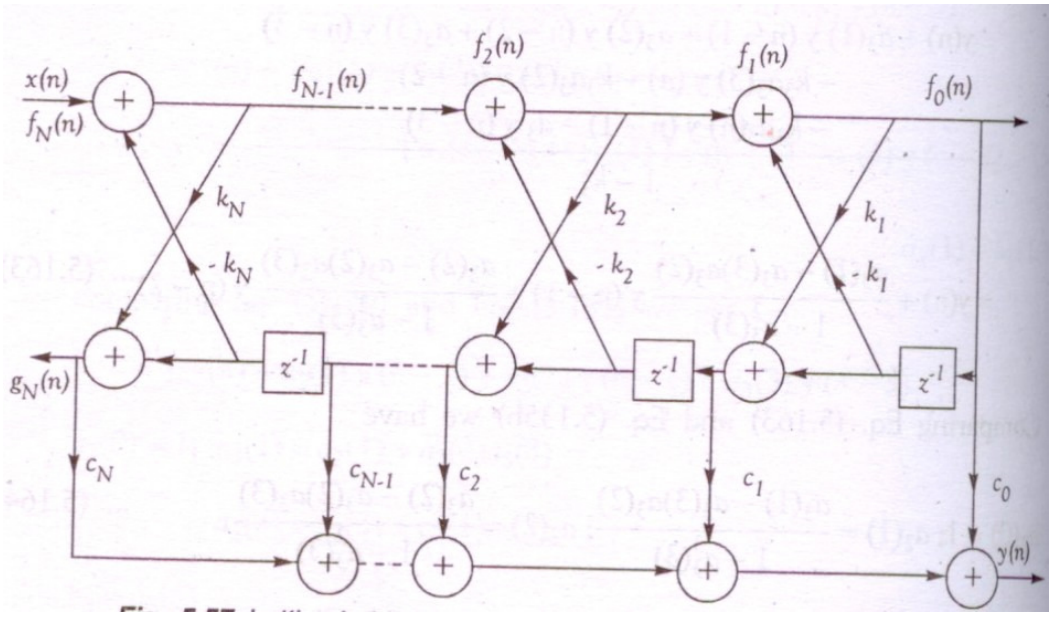
$$H(Z)=\frac{B_M(Z)}{A_N(Z)}=\frac{\sum_{k=0}^{M} b_M(k)Z^{-k}}{1+\sum_{k=1}^{N} a_N(k)Z^{-k}}$$

Where $N \geq M$

A lattice structure can be constructed by first realizing an all-pole lattice co-efficients k_m, $1 \leq m \leq N$ for the denominator $A_N(Z)$, and then adding a ladder part for M=N. The output of the ladder part can be expressed as a weighted linear combination of $\{g_m(n)\}$. Now the output is given by

$$y(n)=\sum_{m=0}^{M} C_m g_m(n)$$

Where $\{C_m\}$ are called the ladder co-efficient and can be obtained using the recursive relation, $C_m = b_m - \sum_{i=m+1}^{M} C_i a_i(i-m)$; m=M, M-1, ….0



Problem:

Convert the following pole-zero IIR filter into a lattice ladder structure,

$$H(Z)=\frac{1+2Z^{-1}+2Z^{-2}+Z^{-3}}{1+\frac{13}{24}Z^{-1}+\frac{5}{8}Z^{-2}+\frac{1}{3}Z^{-3}}$$

Solution:

Given $b_M(Z)=1+2Z^{-1}+2Z^{-2}+Z^{-3}$

And $A_N(Z)=1+\frac{13}{24}Z^{-1}+\frac{5}{8}Z^{-2}+\frac{1}{3}Z^{-3}$

$a_3(0)=1; \quad a_3(1)=\frac{13}{24}; \quad a_3(2)=\frac{5}{8}; \quad a_3(3)=\frac{1}{3}$

$$k_3 = a_3(3) = \tfrac{1}{3}$$

Using the equation

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a^2 m(m)}$$

for m=3, k=1

$$a_2(1) = \frac{a_3(1) - a_3(3)a_3(2)}{1 - a_3^2(3)} = \frac{\frac{13}{24} - \frac{1}{3}\cdot\frac{5}{8}}{1 - \left(\frac{1}{3}\right)^2} = \frac{3}{8}$$

for m=3, & k=2

$$a_2(2) = k_2 = \frac{a_3(2) - a_3(3)a_3(1)}{1 - a_3^2(3)}$$

$$\frac{\frac{5}{8} - \frac{1}{3}\cdot\frac{13}{24}}{1 - \frac{1}{9}} = \frac{\frac{45-13}{72}}{\frac{8}{9}} = \frac{1}{2}$$

for m=2, & k=1

$$a_1(1) = k_1 = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)}$$

$$\frac{\frac{3}{8} - \frac{1}{2}\cdot\frac{3}{8}}{1 - (\frac{1}{2})^2} = \frac{\frac{3}{8} - \frac{3}{16}}{1 - \frac{1}{4}} = \frac{1}{4}$$

for lattice structure $k_1 = \frac{1}{4}, \quad k_2 = \frac{1}{2}, \quad k_3 = \frac{1}{3}$

For ladder structure

$$C_m = b_m - \sum_{i=m+1}^{M} C_1.a_1(1-m) \qquad \text{m=M, M-1,1,0}$$

M=3

$$C_3 = b_3 = 1; \quad C_2 = b_2 - C_3 a_3(1)$$
$$= 2 - 1.(\tfrac{13}{24}) = 1.4583$$

$$C_1 = b_1 - \sum_{i=2}^{3} c_1 a_1(i-m) \qquad \text{m=1}$$

$$= b_1 - c_2 a_2(1) + c_3 a_{3(2)}$$

$$= 2 - \left[(1.4583)(\tfrac{3}{8}) + \tfrac{5}{8}\right] = 0.8281$$

$$c_0 = b_0 - \sum_{i=1}^{3} c_1 a_1(i-m)$$

$$= b_0 - \left[c_1 a_1(1) + c_2 a_2(2) + c_3 a_3(3)\right]$$

$$= 1 - \left[08281(\tfrac{1}{4}) + 1.4583(\tfrac{1}{2}) + \tfrac{1}{3}\right] = -02695$$

To convert a lattice- ladder form into a direct form, we find an equation to obtain $a_N(k)$ from k_m (m=1,2,……N) then equation for c_m is recursively used to compute b_m (m=0,1,2,……M).