

Study of DC transients in R-L and R-C circuits

Objectives

- Definition of inductance and continuity condition for inductors.
- To understand the rise or fall of current in a simple series $R-L$ circuit excited with dc source.
- Meaning of 'Time Constant (τ)' for $R-L$ circuit and explain its relationship to the performance of the circuit.
- Energy stored in an inductor
- Definition of capacitance and Continuity condition for capacitors.
- To understand the rise or fall voltage across the capacitor in a simple series $R-C$ circuit excited with dc source.
- Meaning of 'Time Constant (τ)' for $R-C$ circuit and explain its relationship to the performance of the circuit.
- Energy stored in a capacitor

L.10.1 Introduction

So far we have considered dc resistive network in which currents and voltages were independent of time. More specifically, Voltage (cause \rightarrow input) and current (effect \rightarrow output) responses displayed simultaneously except for a constant multiplicative factor R ($V = R \times I$). Two basic passive elements namely, inductor (L) and capacitor (C) are introduced in the dc network. Automatically, the question will arise whether or not the methods developed in lesson-3 to lesson-8 for resistive circuit analysis are still valid. The voltage/current relationship for these two passive elements are defined by the derivative

(voltage across the inductor $v_L(t) = L \frac{di_L(t)}{dt}$, where $i_L(t)$ = current flowing through the

inductor ; current through the capacitor $i_C(t) = C \frac{dv_C(t)}{dt}$, $v_C(t)$ = voltage across the

capacitor) or in integral form as $i_L(t) = \frac{1}{L} \int_0^t v_L(t) dt + i_L(0)$ or $v_C(t) = \frac{1}{C} \int_0^t i_C(t) dt + v_C(0)$

rather than the algebraic equation ($V = IR$) for all resistors. One can still apply the KCL, KVL, Mesh-current method, Node-voltage method and all network theorems but they result in differential equations rather than the algebraic equations that we have considered in resistive networks (see Lesson-3 to lesson-8).

An electric switch is turned on or off in some circuit (for example in a circuit consisting of resistance and inductance), transient currents or voltages (quickly changing current or voltage) will occur for a short period after these switching actions. After the transient has ended, the current or voltage in question returns to its steady state situation (or normal steady value). Duration of transient phenomena are over after only a few micro or milliseconds, or few seconds or more depending on the values of circuit parameters (like R , L , and C). The situation relating to the sudden application of dc voltage to circuits possessing resistance (R), inductance (L), and capacitance (C) will

now be investigated in this lesson. We will continue our discussion on transients occurring in a dc circuit. It is needless to mention that transients also occur in ac circuit but they are not included in this lesson.

L.10.2 Significance of Inductance of a coil and dc transients in a simple R-L circuit

Fig.10.1 shows a coil of wire forming an inductance and its behavior is to resist any change of electric current through the coil. When an inductor carries current, it produces a certain amount of magnetic flux (Φ) in the core or space around it. The product of the magnetic flux (Φ) and the number of turns of a coil (an inductor) is called the 'flux linkage' of the coil.

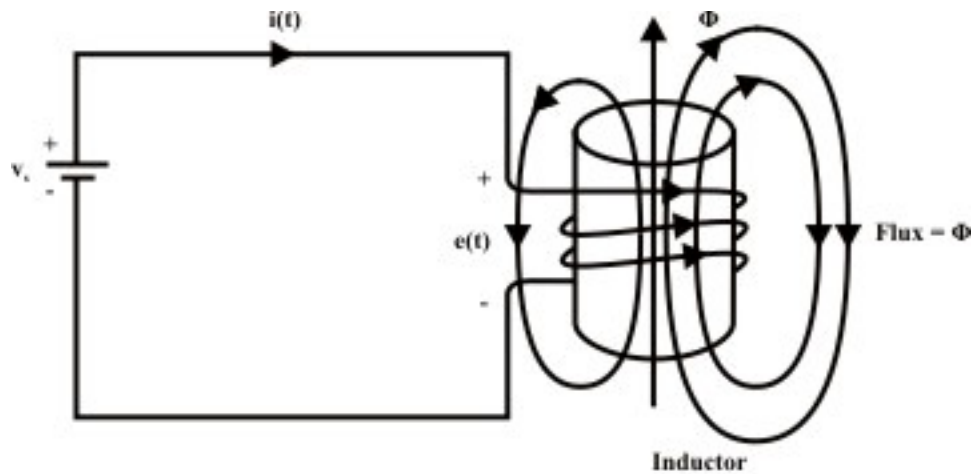


Fig. 10.1

Considering the physical fact that the voltage across the coil is directly proportional to the rate of change of current through the inductor and it is expressed by the equation

$$emf = e(t) = L \frac{di(t)}{dt} \Rightarrow L = \frac{e(t)}{di(t)/dt} \quad (10.1)$$

where L is the constant of proportionality called inductance of coil and it is measured in $\frac{volt - second}{ampere} = henry (H)$. The direction of induced emf is opposite to that of current increases or decreases (Lenz's Law)

$$e(t) = -L \frac{di(t)}{dt} \quad (10.2)$$

Let us assume that the coil of wire has ' N ' turns and the core material has a relatively high permeability (or magnetic path reluctance is very low), so that the magnetic flux (Φ) produced due to current flowing through the coil is concentrated within the core area. The basic fundamental principle according to Faraday, the changing flux through the coil creates an induced emf (e) and it is expressed as

$$e(t) = -N \frac{d\Phi(t)}{dt} \quad (10.3)$$

In words, Faraday’s law states that the voltage induced in a coil (inductor) is proportional to the number of turns that the coil has, and also to the rate of change of the magnetic flux passing through its coils. From equations (10.2) and (10.3), one can write the following relation

$$L = \frac{N d\Phi(t)}{di} = \frac{\text{change in flux linkage}}{\text{change in current}} = \frac{N \Phi}{I} \quad (10.4)$$

The inductance of a coil can also be defined as flux (Φ) linkage per unit of current flowing through the coil and it is illustrated through numerical example.

Example-L.10.1: Consider two coils having the same number of turns ‘ N ’. One coil is wrapped in a nonmagnetic core (say, air) and the other is placed on a core of magnetic material as shown in fig.10.2. Calculate the inductances of both coils for same amount of current flowing through them.

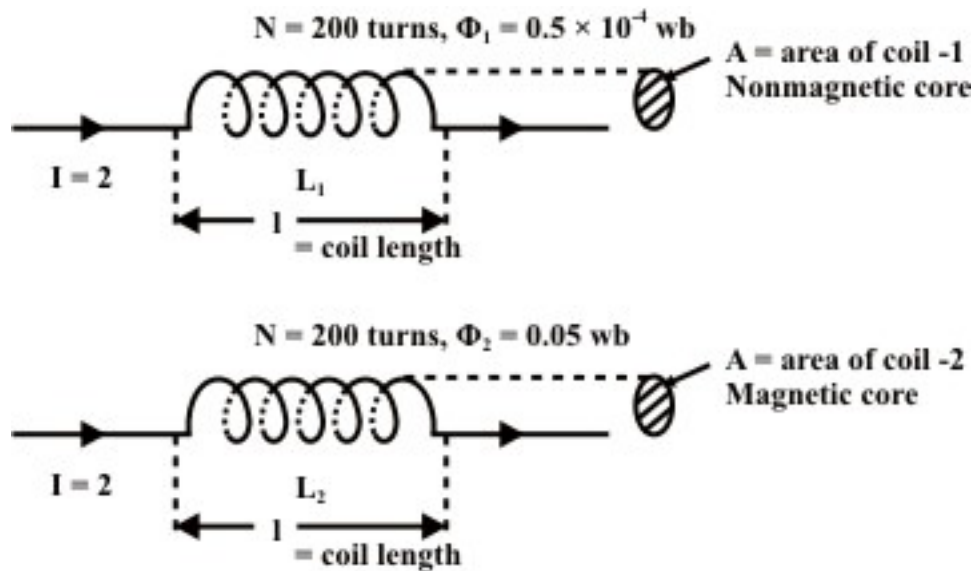


Fig. 10.2: Inductance of coil depends on the surrounding media

Case-A: Nonmagnetic material

$$\text{Inductance of nonmagnetic material} = L_1 = \frac{N \Phi_1}{I} = \frac{200 \times (0.5 \times 10^{-4})}{2} = 5 \text{ mH}$$

Case-B: Magnetic material

$$\text{Inductance of magnetic material} L_2 = \frac{N \times \Phi_2}{I} = \frac{200 \times (0.05)}{2} = 5 \text{ H} \quad (\text{Note: } L_2 > L_1)$$

L.10.2.1 Inductance calculation from physical dimension of coil

A general formula for the inductance of a coil can be found by using an equivalent *Ohm's law* for magnetic circuit and the formula for reluctance. This topic will be discussed in detail in Lesson-21. Consider a solenoid-type electromagnet/toroid

with a length much greater than its diameter (at least the length is ten times as great as its diameter). This will produce an uniform magnetic field inside the toroid. The length ' l ' of a toroid is the distance around the center axis of its core, as indicated in fig.10.3 by dotted line. Its area ' A ' is the cross-sectional area of the toroid, also indicated in that figure.

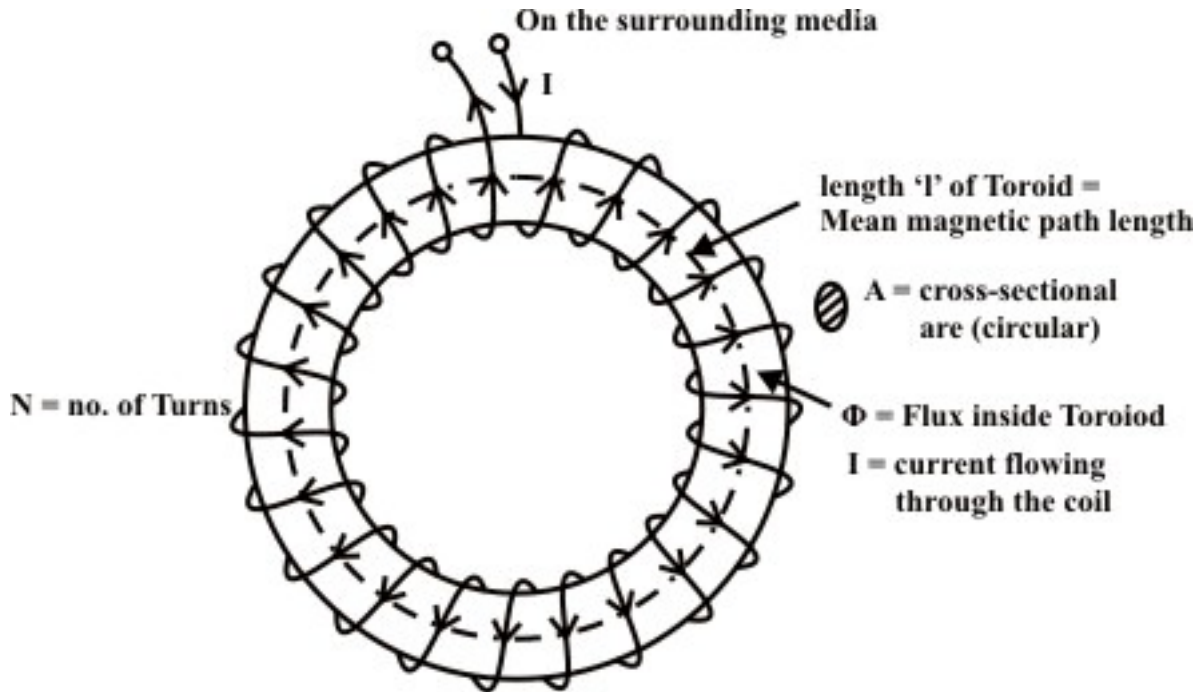


Fig. 10.3: A toroidal inductor

Applying ampere-circuital law for magnetic circuit (see Lesson-21) one can write the following relation

$$NI = Hl \Rightarrow H = \frac{NI}{l} \text{ At/m} \quad (10.5)$$

We know, flux is always given by the product of flux density (B) and area (A) through which flux density is uniform. That is,

$$\Phi = B \times A = \mu H A = \mu \frac{NI}{l} A \quad (\text{note, } \mu = \mu_0 \times \mu_r) \quad (10.6)$$

where $B = \mu H$ and H is the uniform field intensity around the mean magnetic path length ' l '. Substituting the equation (10.6) into the defining equation for inductance, equation (10.4) gives

$$L = \frac{N\Phi}{I} = \frac{\mu N^2 I A}{Il} = \frac{\mu N^2 A}{l} \quad (10.7)$$

Remark-1: The expression (10.7) is derived for long solenoids and toroids, computation of inductance is valid only for those types.

L.10.2.2 Continuity condition of Inductors

The current that flows through a linear inductor must always be a continuous. From the expression (10.1), the voltage across the inductor is not proportional to the current flowing through it but to the rate of change of the current with respect to time, $\frac{di(t)}{dt}$. The voltage across the inductor (v_L) is zero when the current flowing through an inductor does not change with time. This observation implies that the inductor acts as a short circuit under steady state dc current. In other words, under the steady state condition, the inductor terminals are shorted through a conducting wire. Alternating current (ac), on the other hand, is constantly changing; therefore, an inductor will create an opposition voltage polarity that tends to limit the changing current. If current changes very rapidly with time, then inductor causes a large opposition voltage across its terminals. If current changes through the inductor from one level to another level instantaneously i.e. in $dt \approx 0$ sec., then the voltage across it would become infinite and this would require infinite power at the terminals of the inductor. Thus, instantaneous changes in the current through an inductor are not possible at all in practice.

Remark-2: (i) The current flowing through the inductor cannot change instantaneously (i.e. $i(0^-)$ just right before the change of current = $i(0^+)$ just right after the change of current). However, the voltage across an inductor can change abruptly. (ii) The inductor acts as a short circuit (i.e. inductor terminals are shorted with a conducting wire) when the current flowing through the inductor does not change (constant). (iii) These properties of inductor are important since they will be used to determine “boundary conditions”.

L.10.3 Study of dc transients and steady state response of a series R-L circuit.

Ideal Inductor: Fig.10.4 shows an ideal inductor, like an ideal voltage source, has no resistance and it is excited by a dc voltage source V_s .

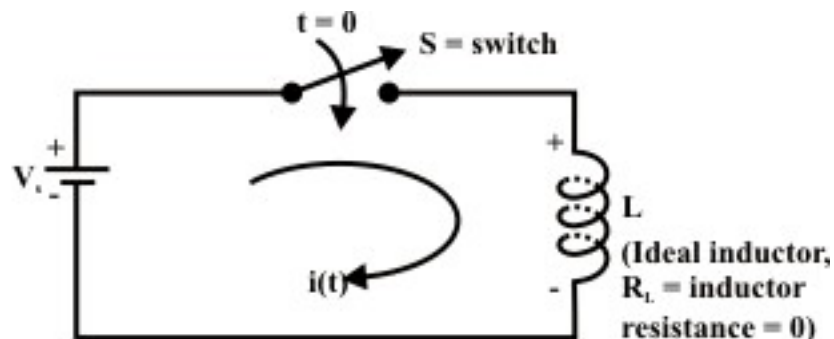


Fig. 10.4: An ideal inductor connected to a constant voltage source

The switch 'S' is closed at time 't = 0' and assumed that the initial current flowing through the ideal inductor $i(0)$ just before closing the switch is equal to zero. To find the system response ($i(t) - v_s - t$), one can apply KVL around the closed path.

KVL

$$V_s - L \frac{di(t)}{dt} = 0 \Rightarrow \frac{di(t)}{dt} = \frac{V_s}{L} \tag{10.8}$$

$$\int_0^{i(t)} di(t) = \frac{V_s}{L} \int_0^t dt \Rightarrow i(t) = \frac{V_s}{L} t + i(0) \Rightarrow i(t) = \frac{V_s}{L} t \quad (\text{note } i(0) = 0) \tag{10.9}$$

Equation (10.9) implies that the current through inductor increases with increase in time and theoretically it approaches to infinity as $t \rightarrow \infty$ but in practice, this is not really the case.

Real or Practical inductor:

Fig.10.5 shows a real or practical inductor has some resistance and it is exactly equal to the resistance of the wire used to wind the coil.

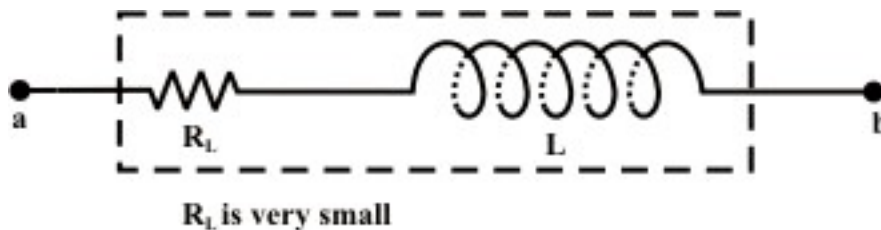


Fig. 10.5: Representation of an practical inductor

Let us consider a practical inductor is connected in series with an external resistance R_1 and this circuit is excited with a dc voltage V_s as shown in fig.10.6(a).

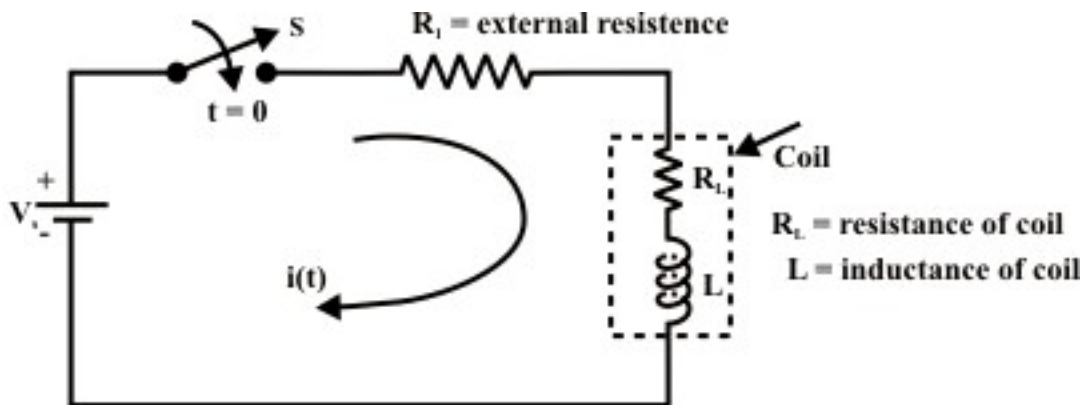


Fig. 10.6(a): a practical inductor connected to a constant voltage source

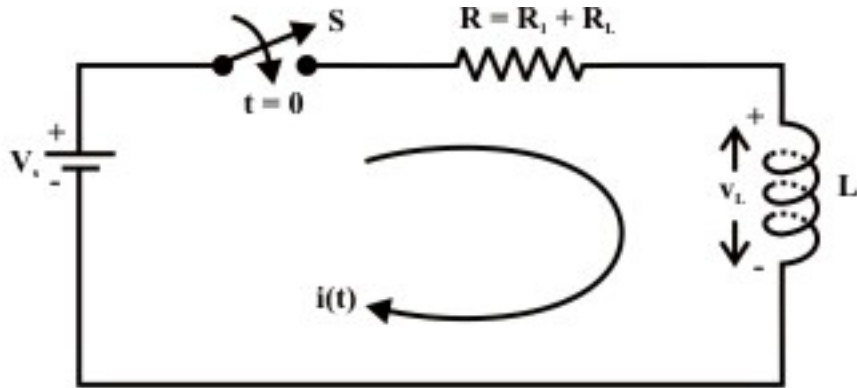


Fig. 10.6(b): Equivalent representation circuit shown in fig. L.10.6(a)

Our problem is to study the growth of current in the circuit through two stages, namely; (i) dc transient response (ii) steady state response of the system.

D.C Transients: The behavior of the current ($i(t)$); charge ($q(t)$) and the voltage ($v(t)$) in the circuit (like $R-L$; $R-C$; $R-L-C$ circuit) from the time ($t(0^+)$) switch is closed until it reaches its final value is called dc transient response of the concerned circuit. The response of a circuit (containing resistances, inductances, capacitors and switches) due to sudden application of voltage or current is called transient response. The most common instance of a transient response in a circuit occurs when a switch is turned on or off – a rather common event in an electric circuit.

L.10.3.1 Growth or Rise of current in R-L circuit

To find the current expression (response) for the circuit shown in fig. 10.6(a), we can write the KVL equation around the circuit

$$V_s - (R_1 + R_L)i(t) - v_L(t) = 0 \Rightarrow V_s = R i(t) + L \frac{di(t)}{dt} \quad (10.10)$$

where V_s is the applied voltage or forcing function, R_L is the resistance of the coil, R_1 is the external resistance. One can combine the resistance of coil R_L to the external resistance R_1 in order to obtain a simplified form of differential equation. The circuit configuration shown in fig. 10.6(a) is redrawn equivalently in fig.10.6(b) for our convenience. The equation (10.10) is the standard first order differential equation and its solution can be obtained by classical method. The solution of first or second order differential equation is briefly discussed in Appendix (at the end of this lesson-10). The following relation gives the solution of equation (10.10)

$$i(t) = i_n(t) + i_f(t) = A_1 e^{\alpha t} + A \quad (10.11)$$

Here, $i_n(t)$ is the complementary solution/natural solution of differential equation (10.10). It is also sometimes called as transient response of system (i.e. the first part of

response is due to an initial condition of the system or force free response). The second part $i_f(t)$ of eq. (10.11) is the particular integral solution/force response or steady state response of the system due to the forcing function ($f(t)=V_s$) or input signal to the series $R-L$ circuit. It may be noted the term A provide us the steady state solution of the first order differential equation while the forcing function (or input to the system) is step function (or constant input). More specifically, for a linear system, the steady state solution of any order differential equation is the same nature of forcing function ($f(t)$) or input signal but different in magnitude. We have listed in tabular form the nature of steady state solution of any order differential equation for various types of forcing functions (see in Appendix). To get the complete solution of eq. (10.10), the values of α , A_1 and A are to be computed following the steps given below:

Step-1: How to find the value of α ?

Assigning $V_s = 0$ and introducing an operator $\alpha = \frac{d}{dt}$ in eq.(10.10) , we get a characteristic equation that will provide us the numerical value of α . This in turn, gives us the transient response of the system provided the constant A_1 is known to us.

The Characteristic equation of (10.10) is $R + \alpha L = 0 \Rightarrow \alpha = -\frac{R}{L}$.

Step-2: How to obtain the constants A_1 and A ?

It may be noted that the differential eq. (10.10) must be satisfied by the particular integral solution or steady state solution $i_f(t)$. The value of $i_f(t)$ at steady state condition (i.e. $t \rightarrow \infty$) can be found out using the eq.(10.11) and it is given below.

Using final condition ($t \rightarrow \infty$)

$$V_s = R i_f(t) + L \frac{d i_f(t)}{dt} \quad (10.12)$$

(note: at steady state ($t \rightarrow \infty$) $i_f(t) = A = \text{const}$ from eq. (10.11))

$$V_s = R A + L \frac{d A}{dt} \Rightarrow A = \frac{V_s}{R} \quad (10.13)$$

Using initial condition ($t = 0$)

Case-A: Assume current flowing through the inductor just before closing the switch ‘S’ (at $t=0^-$) is $i(0^-)=0$.

$$i(0) = i(0^-) = i(0^+) = A_1 + A \quad (10.14)$$

$$0 = A_1 + A \Rightarrow A_1 = -A = -\frac{V_s}{R}$$

Using the values of α , A_1 and A in equation (10.11), we get the current expression as

$$i(t) = \frac{V_s}{R} \left[1 - e^{-\frac{R}{L}t} \right] \quad (10.15)$$

The table shows how the current $i(t)$ builds up in a R-L circuit.

Actual time (t) in sec	Growth of current in inductor (Eq.10.15)
$t = 0$	$i(0) = 0$
$t = \tau \left(= \frac{L}{R} \right)$	$i(\tau) = 0.632 \times \frac{V_s}{R}$
$t = 2\tau$	$i(2\tau) = 0.865 \times \frac{V_s}{R}$
$t = 3\tau$	$i(3\tau) = 0.950 \times \frac{V_s}{R}$
$t = 4\tau$	$i(4\tau) = 0.982 \times \frac{V_s}{R}$
$t = 5\tau$	$i(5\tau) = 0.993 \times \frac{V_s}{R}$

Note: Theoretically at time $t \rightarrow \infty$ the current in inductor reaches its steady state value but in practice the inductor current reaches 99.3% of its steady state value at time $t = 5\tau$ (sec.).

The expression for voltage across the external resistance R_1 (see Fig. 10.6(a))

$$= v_{R_1} = i(t) R_1 = \frac{V_s}{R} \left[1 - e^{-\frac{R}{L}t} \right] R_1 \quad (10.16)$$

The expression for voltage across the inductor or coil

$$v_{coil}(t) = v_{inductor}(t) = V_s - v_{R_1}(t) = V_s - \frac{V_s}{R} (R_1) \left[1 - e^{-\frac{R}{L}t} \right] \quad (10.17)$$

Graphical representation of equations (10.15)-(10.17) are shown in Fig.10.7 for different choices of circuit parameters (i.e., L & R)

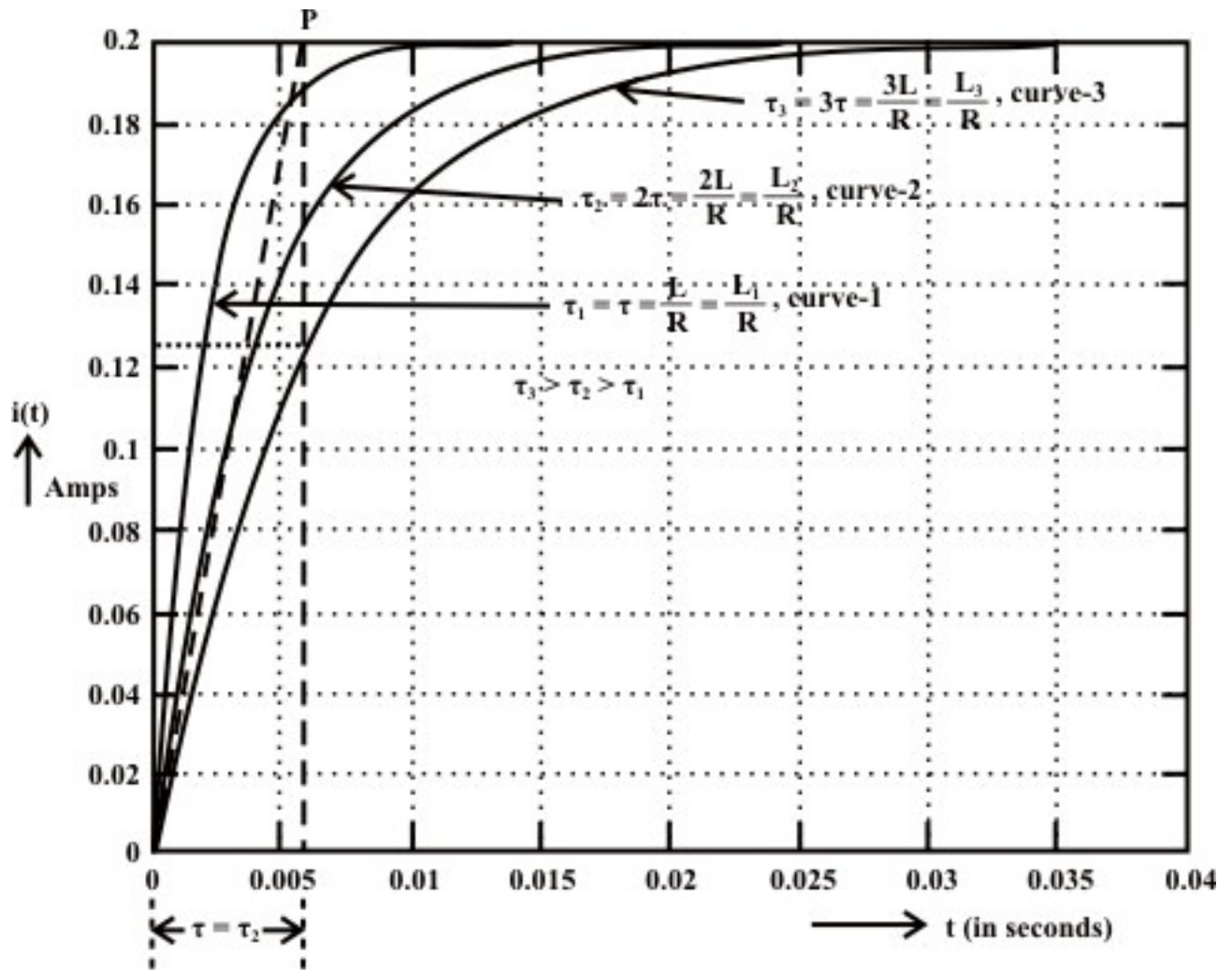


Fig. 10.7(a): Growth of current in R-L circuit (assumed initial current through inductor is zero).

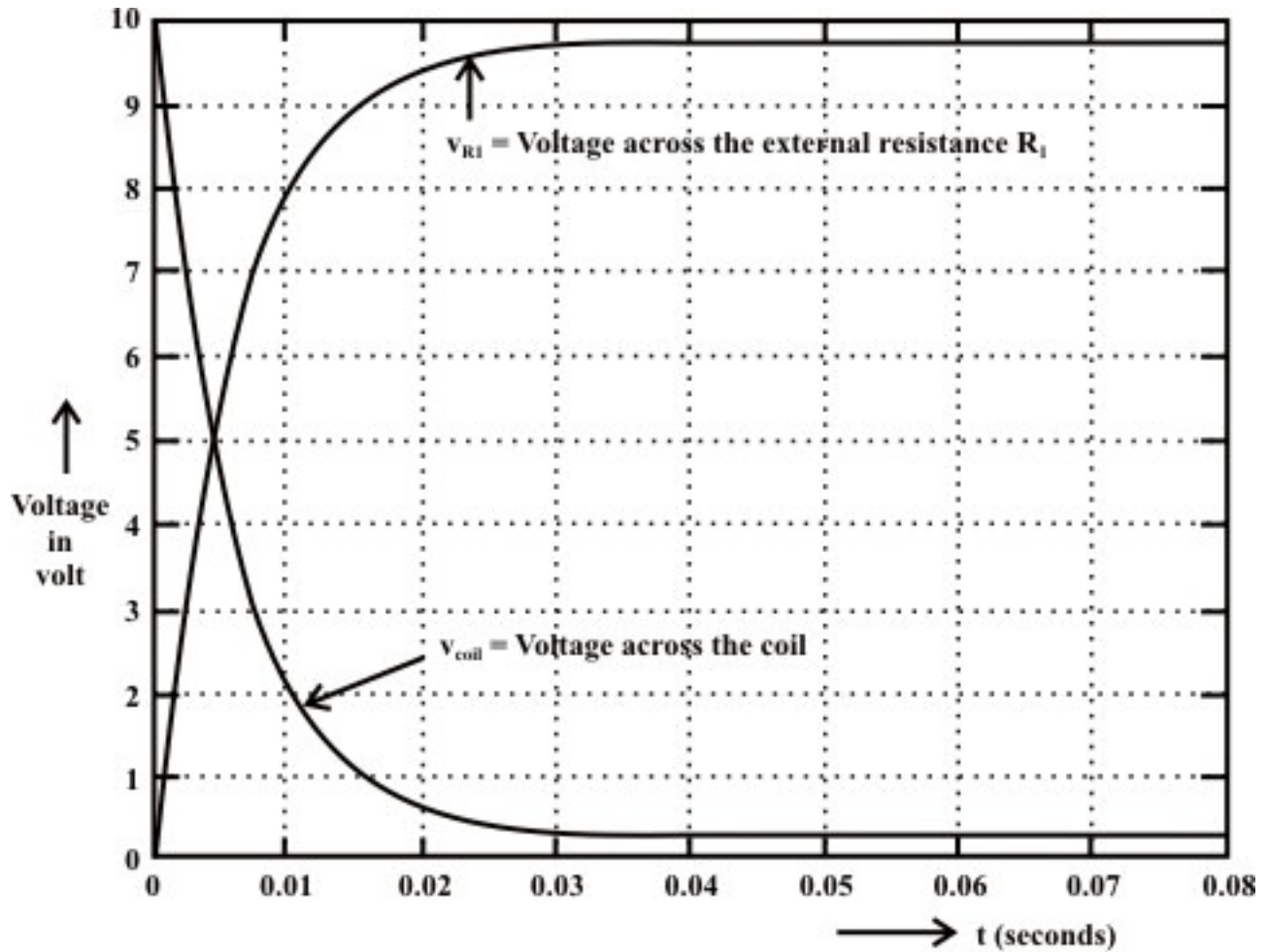


Fig. 10.7(b): Voltage response in different elements of R-L circuit (assumed $i_s = 0$)

Case-B: Assume current flowing through the inductor just before closing the switch ‘S’ (at $t=0^-$) is $i(0^-)=i_0 \neq 0$.

Using equations (10.13) and (10.14), we get the values of $A = \frac{V_s}{R}$ and $A_1 = i(0) - A = i_0 - \frac{V_s}{R}$. Using these values in equation (10.11), the expression for current flowing through the circuit is given by

$$i(t) = \frac{V_s}{R} \left[1 - e^{-\frac{R}{L}t} \right] + \left(i_0 e^{-\frac{R}{L}t} \right) \quad (10.18)$$

The second part of the right hand side of the expression (10.18) indicates the current flowing to the circuit due to initial current i_0 of inductor and the first part due to the forcing function V_s applied to the circuit. This means that the complete response of the circuit is the algebraic sum of two outputs due to two inputs; namely (i) due to forcing function V_s (ii) due to initial current i_0 through the inductor. This implies that the superposition theorem is also valid for such type of linear circuit. Fig.10.8 shows the

response of inductor current when the circuit is excited with a constant voltage source V_s and the initial current through inductor is i_0 .

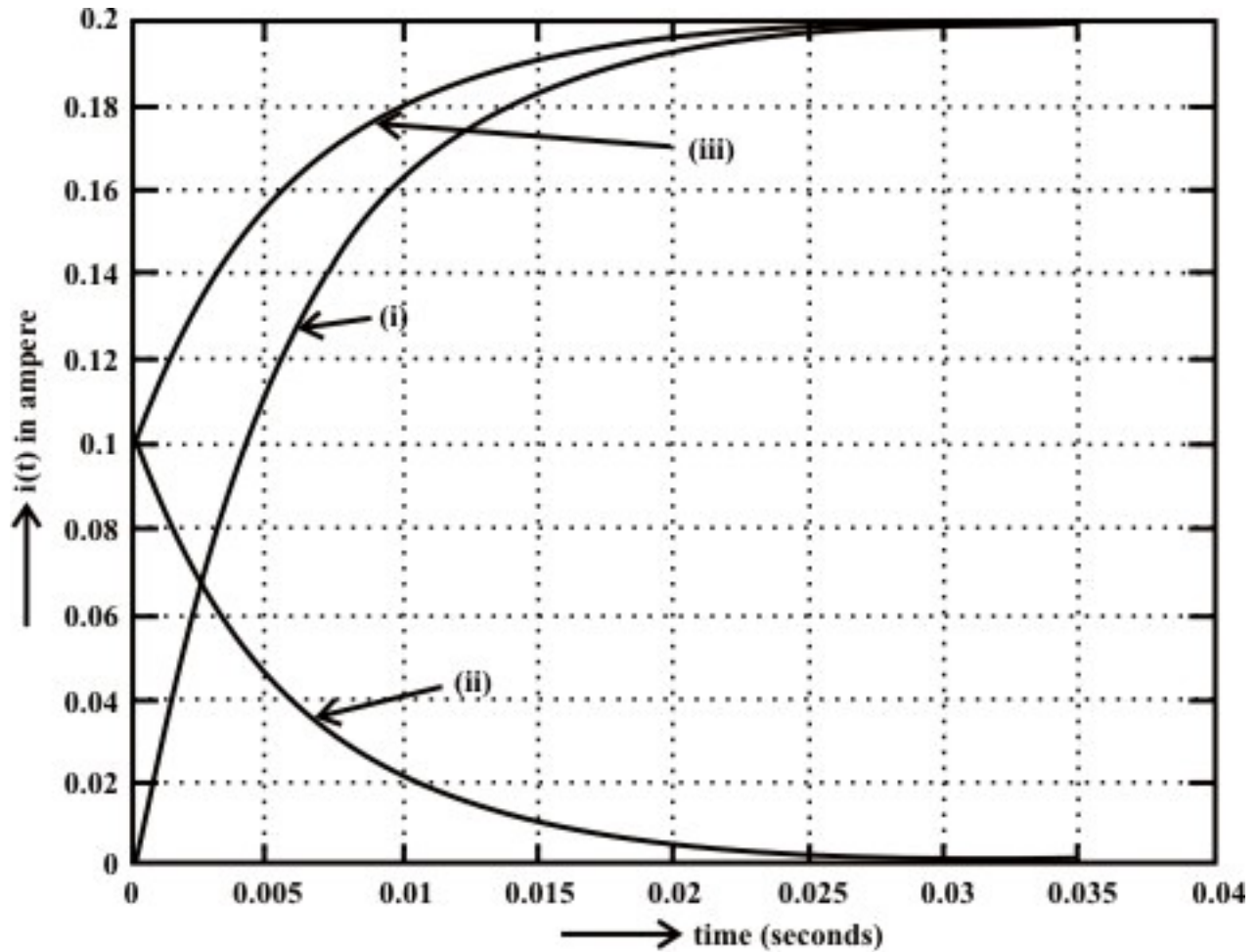


Fig. 10.8: Current through inductor due to (i) forcing function V_s only, (ii) initial condition i_0 only, (iii) combined effect of (i) and (ii)

Remark-3: One can also solve this differential equation by separating the variables and integrating.

Time constant (τ) for exponential growth response (τ): We have seen that the current through inductor is represented by

$$i(t) = \frac{V_s}{R} \left[1 - e^{-\frac{R}{L}t} \right]$$

when a series $R-L$ circuit is excited by a constant voltage source (V_s) and an initial current through the inductor i_0 is assumed to be zero. Further it may be noted that the current through the inductor (see fig.10.7) increases as time increases. The shape of

growing current before it reaches to a steady state value $\left(\frac{V_s}{R}\right)$ entirely depends on the parameters of $R-L$ circuit (i.e. R & L) that associated with the exponential term $\left(e^{-\frac{R}{L}t}\right)$. As 't' grows larger and larger the transient, because of its negative exponential factor, diminishes and disappears, leaving only the steady state.

Definition of Time Constant (τ) of $R-L$ Circuit: It is the time required for any variable or signal (in our case either current ($i(t)$) or voltage (v_{Rl} or v_L)) to reach 63.2% (i.e the time at which the factor $\left[1 - e^{-\frac{R}{L}t}\right] \times 100$ in eq.(10.15) becomes $(1 - e^{-1}) \times 100 = 63.2\%$) of its final value. It is possible to write an exact mathematical expression to calculate the time constant (τ) of any first-order differential equation.

Let 't' is the time required to reach 63.2% of steady-state value of inductor current (see fig. 10.6(a)) and the corresponding time 't' expression can be obtained as

$$i(t) = 0.632 * \frac{V_s}{R} = \frac{V_s}{R} \left[1 - e^{-\frac{R}{L}t} \right] \Rightarrow 0.632 = 1 - e^{-\frac{R}{L}t} \Rightarrow 0.368 = e^{-\frac{R}{L}t} = e^{-1} \Rightarrow t =$$

$$\frac{L}{R} = \tau (\text{sec.})$$

The behavior of all circuit responses (for first-order differential equation) is fixed by a single time constant τ (for $R-L$ circuit $\tau = \frac{L}{R}$) and it provides information about the speed of response or in other words, it indicates how first or slow the system response reaches its steady state from the instant of switching the circuit. Observe the equation (10.15) that the smaller the time constant (τ), the more rapidly the current increases and subsequently it reaches the steady state (or final value) quickly. On the other hand, a circuit with a larger time constant (τ) provides a slow response because it takes longer time to reach steady state. These facts are illustrated in fig.10.7(a). In accordance with convenience, the time constant of an exponential term (say $p(t) = p_0(1 - e^{-at})$) is the reciprocal of the coefficient 'a' associated with the 't' in the power of exponential term.

Remark-4: An interesting property of exponential term is shown in fig. 10.7(a). The time constant τ of a first order differential equation may be found graphically from the response curve. It is necessary to draw a tangent to the exponential curve at time 't=0' and maintained the same slope until it intersects the steady state value of current curve at P point. A perpendicular is drawn from the point P to the time axis and it intersects the time axis at $t = \tau$ (see fig. 10.7(a)). Mathematically, this can be easily verified by considering the equation of a straight line tangent to the current curve at $t = 0$, given by $y = mt$ where m is the slope of the straight line, expressed as

$$m = \frac{di(t)}{dt} \Big|_{t=0} = \frac{d\left(\frac{V_s}{R} \left[1 - e^{-\frac{R}{L}t}\right]\right)}{dt} \Big|_{t=0} = \frac{V_s}{L} \quad (10.19)$$

Here, we designated the value of time 't' required to reach y from '0' to $\frac{V_s}{R}$ units, assuming a constant rate (slope) of growth. Thus,

$$\frac{V_s}{R} = \frac{V_s}{L}t \Rightarrow t = \frac{L}{R} = \tau \text{ (sec.)} \quad (10.20)$$

It is often convenient way of approximating the time constant (τ) of a circuit from the response curve (see fig.10.7(a) for curve-2).

L.10.3.2 Fall or Decay of current in a R-L circuit

Let us consider the circuit shown in fig. 10.9(a). In this circuit, the switch 'S' is closed sufficiently long duration in position '1'. This means that the current through the inductor reaching its steady-state value ($I = \frac{V_s}{R} = \frac{V_s}{R_1 + R_L} = I_0$) and it acts, as a short circuit i.e. the

voltage across the inductor is nearly equal to zero since resistance $R_L \ll R_1$. If the switch 'S' is opened at time 't'=0 and kept in position '2' for $t > 0$ as shown in fig. 10.9(b), this situation is referred to as a source free circuit.

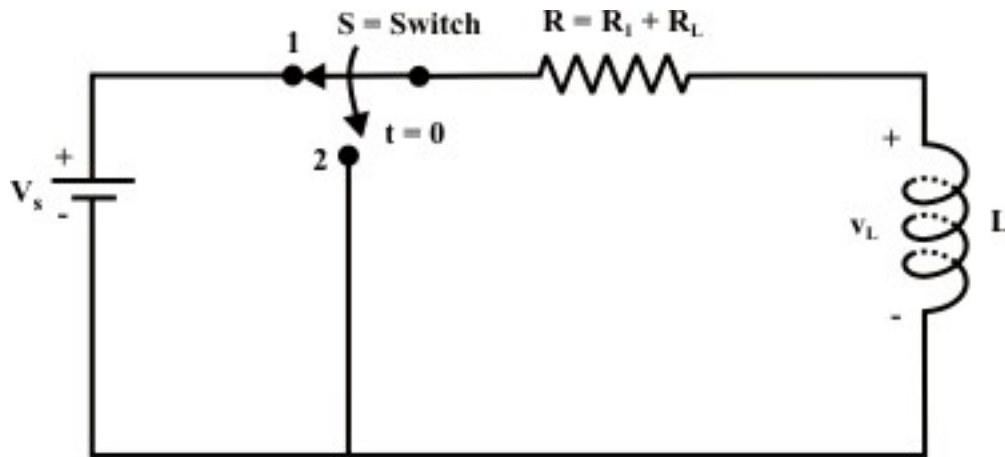


Fig. 10.9(a)

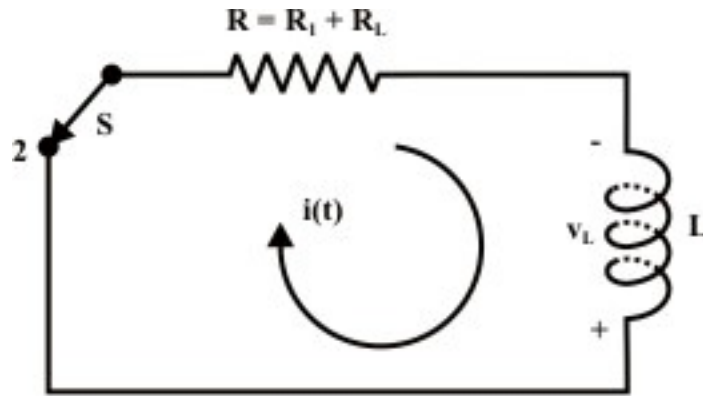


Fig. 10.9(b): Decay of current in R - L circuit

Since the current through an inductor cannot change instantaneously, the current through the inductor just before ($i(0^-)$) and after ($i(0^+)$) opening the switch 'S' must be same. Because there is no source to sustain the current flow in inductor, the magnetic field in inductor starts to collapse and this, in turn, will induce a voltage across the inductor. The polarity of this induced voltage across the inductor is just in reverse direction compared to the situation that occurred during the growth of current in inductor (i.e. when the switch 'S' is kept in position '1'). This is illustrated in fig. 10.9(b), where the voltage induced in inductor is positive at the bottom of the inductor terminal and negative at the top. This implies that the current through inductor will still flow in the same direction, but with a magnitude decaying toward zero. Applying KVL around the closed circuit in fig. 10.9(b), we obtain

$$L \frac{di(t)}{dt} + Ri(t) = 0 \quad (10.21)$$

The solution of the homogeneous (input free), first-order differential equation with constant coefficients subject to the initial (boundary) inductor current (initial condition, $i(0^-) = i(0^+) = \frac{V_s}{R} = I$) is given by

$$i(t) = i_n(t) = A_1 e^{\alpha t} \quad (10.22)$$

where α can be found from the characteristic equation of eq.(10.21) described by

$$L\alpha + R = 0 \Rightarrow \alpha = -\frac{R}{L} \quad (10.23)$$

At time $t=0$, the initial condition $i(0^-) = i(0^+) = \frac{V_s}{R}$ is used in equation (10.22) to compute the constant A_1 and it is given below.

$$i(0) = A_1 \Rightarrow A_1 = \frac{V_s}{R} = I$$

Using the values of A_1 and α in equation (10.22), we get final expression as

$$i(t) = \frac{V_S}{R} e^{-\frac{R}{L}t} \quad \text{for } t \geq 0 \quad (10.24)$$

A sketch of $i(t)$ for $t \geq 0$ is shown in fig.10.10. Here, transient has ended and steady state has been reached when both current in inductor $i(t)$ and voltage across the inductor including its internal resistance are zero.

Time Constant (τ) for exponential decay response: For the source free circuit, it is the time τ by which the current falls to 36.8 percent of its initial value. The initial condition in this case (see fig. 10.9(a)) is considered to be the value of inductor's current at the moment the switch S is opened and kept in position '2'. Mathematically, τ is computed as

$$i(t) = 0.368 \times \frac{V_S}{R} = \frac{V_S}{R} e^{-\frac{R}{L}t} \Rightarrow t = \tau = \frac{L}{R} \quad (10.25)$$

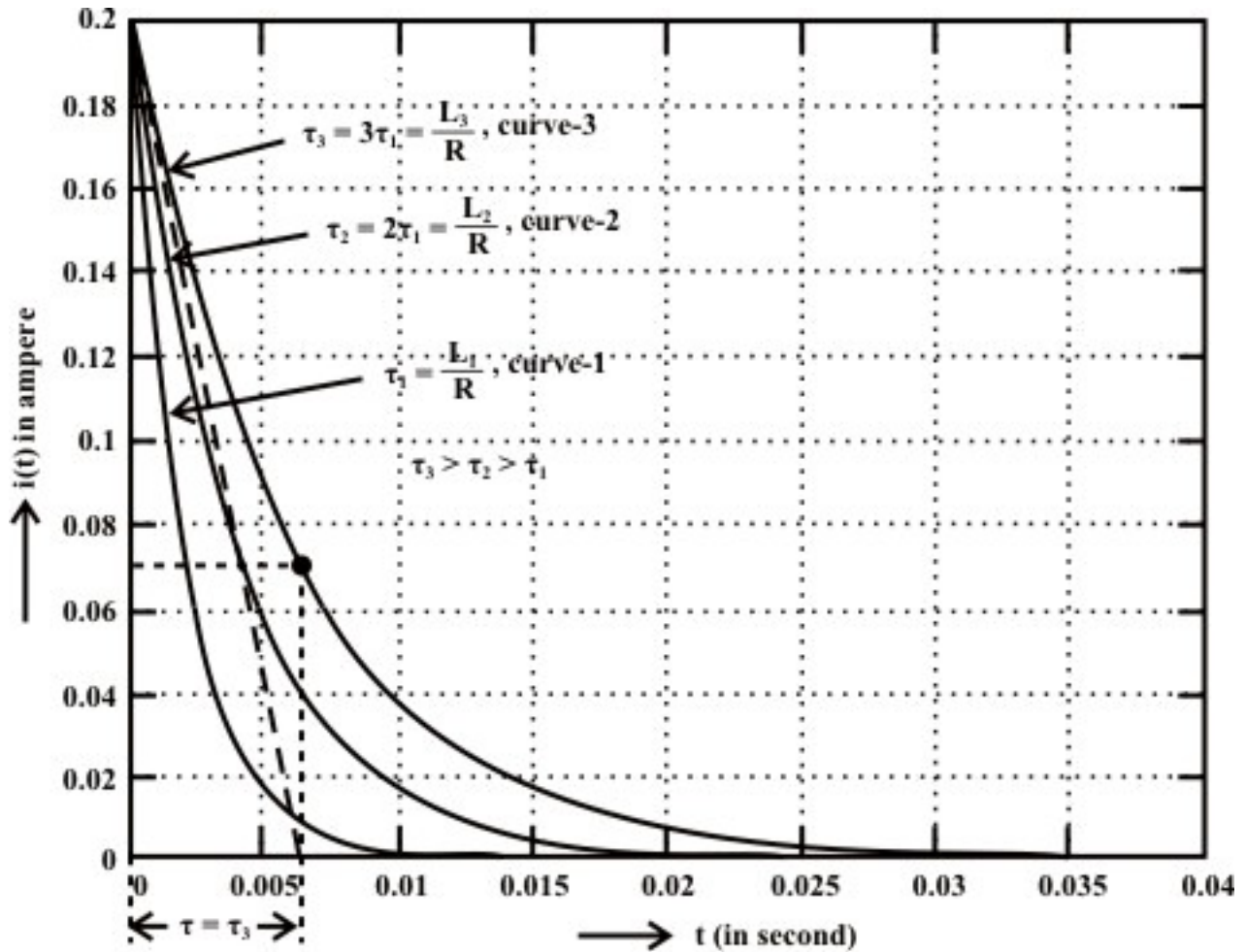


Fig. 10.10: Fall of current in R-L circuit (assumed initial current through inductor is I).

Alternatively, the time constant for an exponential decay response of a circuit may be computed graphically by adopting the steps (see equations (10.19) and (10.20)) as discussed before. In fig.10.10, a tangent is drawn to the exponential decay curve at time 't=0' and maintained the same slope until the straight line intercepts time axis at time $t=\tau$. Approximately, the value of τ can thus be found directly from graphical representation of exponential decay curve.

L.10.3.3 Energy stored in an inductor

Let us turn our attention to power and energy consideration for an inductor. The instantaneous power absorbed by the inductor is expressed by product of the current through inductor $i(t)$ and the voltage across it $v(t)$.

$$p(t) = v(t)i(t) = i(t)L \frac{di(t)}{dt} \tag{10.26}$$

Since the energy is the product of power and time, the energy absorbed by an inductor over a period is expressed as

$$W_L = \int_{t_0}^t p(t) dt = \int_{t_0}^t i(t)L \frac{di(t)}{dt} dt = \frac{1}{2} L [i^2(t) - i^2(t_0)] \quad (10.27)$$

where we select the current through inductor at time ' $t_0 = -\infty$ ' is $i(-\infty) = 0$. Then, we have $W_L = \frac{1}{2} L i^2(t)$ and from this relation we see that the energy stored in an inductor is always non-negative. At any consequent time at which the current is zero, no energy is stored in the inductor. The ideal inductor ($R_L = 0\Omega$) never dissipates energy, but only stores. In true sense, a physical or practical inductor dissipates a very small amount of stored energy due to its small series resistance.

Example-L.10.2 Fig.10.11 shows the plot of current $i(t)$ through a series $R-L$ circuit when a constant forcing function of magnitude $V_s = 50 V$ is applied to it. Calculate the values of resistance R and inductance L .

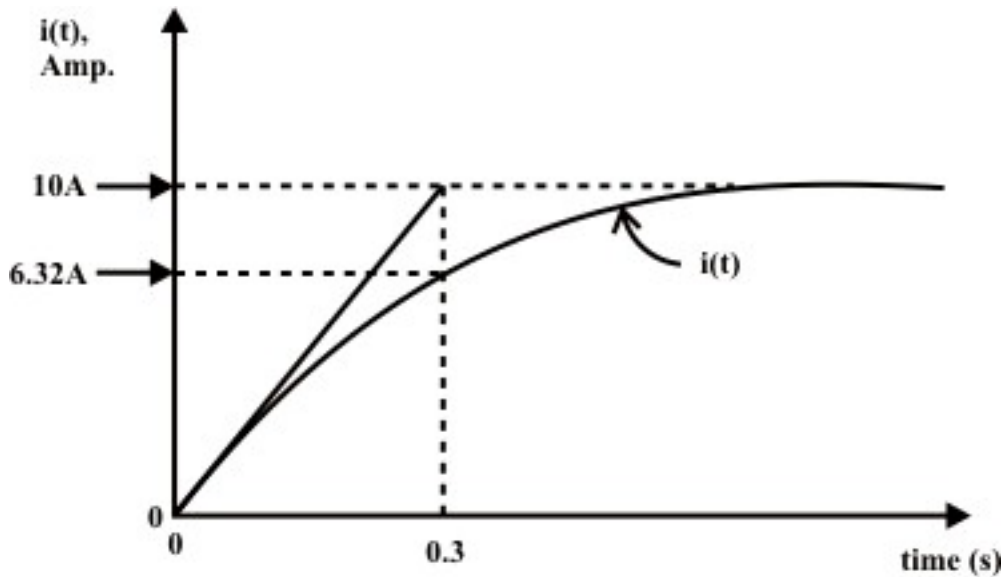


Fig. 10.11: Current -y_s-time characteristic

Solution: From fig.10.11 one can easily see that the steady state current flowing through the circuit is 10 A and the time constant of the circuit $\tau = 0.3$ sec. The following relationships can be written as

$$i_{steady\ state} = \frac{V_s}{R} \Rightarrow 10 = \frac{50}{R} \Rightarrow R = 5 \Omega$$

$$\text{and } \tau = \frac{L}{R} \Rightarrow 0.3 = \frac{L}{5} \Rightarrow L = 1.5 H$$

Example-L.10.3 For the circuit shown in Fig.10.12, the switch ' S ' has been closed for a long time and then opens at $t = 0$.

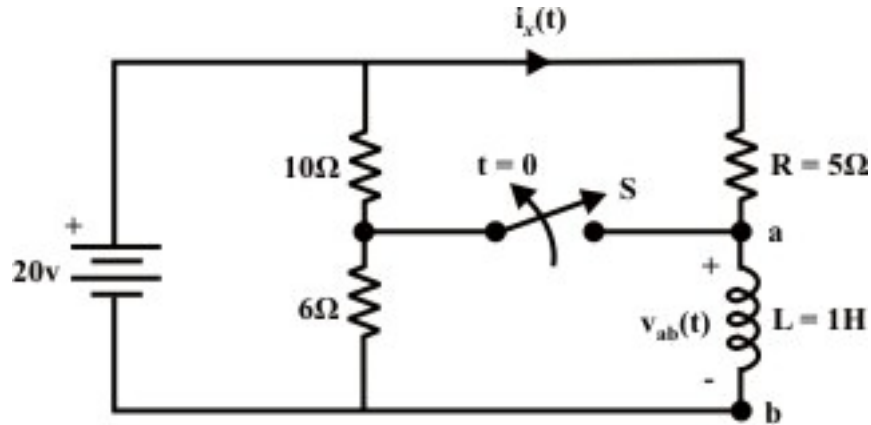


Fig. 10.12

Find,

- (i) $v_{ab}(0^-)$ (ii) $i_x(0^-), i_L(0^-)$ (iii) $i_x(0^+)$ (iv) $v_{ab}(0^+)$ (v) $i_x(t=\infty)$ (vi) $v_{ab}(t=\infty)$
 (vii) $i_x(t)$ for $t > 0$

Solution: When the switch S was in closed position for a long time, the circuit reached in steady state condition i.e. the current through inductor is constant and hence, the voltage across the inductor terminals a and b is zero or in other words, inductor acts as short circuit i.e., (i) $v_{ab}(0^-) = 0V$. It can be seen that the no current is flowing through 6Ω resistor. The following are the currents through different branches just before the switch 'S' is opened i.e., at $t = 0^-$.

$i_x(0^-) = \frac{20}{5} = 4A$ and the current through 10Ω resistor, $i_{10\Omega}(0^-) = \frac{20}{10} = 2A$. The algebraic sum of these two currents is flowing through the inductor i.e., (ii) $i_L(0^-) = 2 + 4 = 6A$.

When the switch 'S' is in open position

The current through inductor at time $t = 0^+$ is same as that of current $i_L(0^-)$, since inductor cannot change its current instantaneously. Therefore, the current through $i_x(0^+)$ is given by

$$i_x(0^+) = i_L(0^+) = 6A.$$

Applying KVL around the closed loop at $t = 0^+$ we get,

$$20 - i_x(0^+) \times R = v_{ab}(0^+) \Rightarrow 20 - 6 \times 5 = v_{ab}(0^+) \Rightarrow v_{ab}(0^+) = -10V$$

The negative sign indicates that inductor terminal 'b' as +ve terminal and it acts as a source of energy or mathematically, $v_{ba}(0^+) = 10V$.

At steady state condition ($t \rightarrow \infty$) the current through inductor is constant and hence inductor acts as a short circuit. This establishes the following relations:

$$v_{ba}(t = \infty) = 0V \quad \text{and} \quad i_x(t = \infty) = \frac{20}{5} = 4A \tag{10.28}$$

The circuit expression $i_x(t)$ for $t \geq 0$ can be obtained using the KVL around the closed path (see fig.10.12).

KVL equation:

$$V_s - i_x(t) \times 5 - L \frac{di_x(t)}{dt} = 0$$

$$i_x(t) \times 5 + L \frac{di_x(t)}{dt} = V_s \quad (10.29)$$

The solution of first order differential equation due to forcing function and initial condition is given by

$$i_x(t) = A_1 e^{-\frac{R}{L}t} + A \quad (10.30)$$

Initial and final conditions are: (i) At $t=0$, $i_x(0) = i_L(0^-) = i_L(0^+) = 6\text{ A}$ (ii) $t \rightarrow \infty$, current through inductor $i_L(t = \infty) = 4\text{ A}$ (see Eq. 10.28). Using initial and final conditions equation (10.30) we get, $A_1 = 6 - A$ and $A = 4 \Rightarrow A_1 = 2$

From equation (10.30), we get the final expression as $i_x(t) = 4 + 2e^{-\frac{5}{1}t}$ for $t \geq 0$.

Example: L.10.4 The switch 'S' is closed in position '1' sufficiently long time and then it is kept in position '2' as shown in fig.10.13. Compute the value of v_L and i_L (i) the instant just prior to the switch changing; (ii) the instant just after the switch changes. Find also the rate of change of current through the inductor at time $t = 0^+$ (i.e., $\left. \frac{di_L(t)}{dt} \right|_{t=0^+}$).

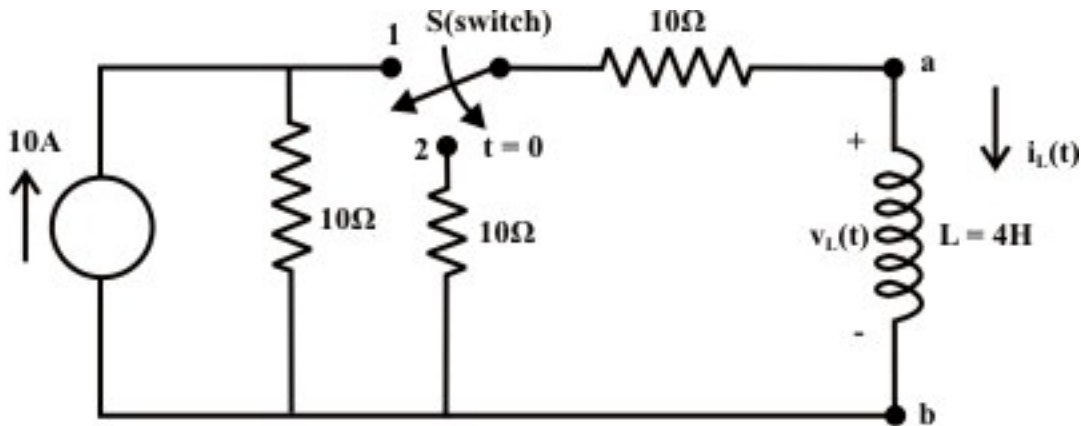


Fig. 10.13

Solution: We assume that the circuit has reached at steady state condition when the switch was in position '1'. Note, at steady state the inductor acts as short circuit and voltage across the inductor is zero.

At $t=0^-$, the current through and the voltage across the inductor are $i_L(0^-) = \frac{10}{10+10} \times 10 = 5A$ and $v_L(0^-) = 0V$ respectively. When the switch is kept in position '2', current through the inductor cannot change instantaneously but this is not true for the voltage across the inductor. At $t=0^+$, one can write the following expressions:

$i_L(0^+) = 5A$ and $v_L(0^+) = -(10 + 10) \times 5 = -100V$ ('b' is more +ve potential than 'a' terminal). Note that the stored energy in inductor is dissipated in the resistors. Now, the rate of change of current through inductor at time $t=0^+$ is obtained as

$$L \left. \frac{di_L(t)}{dt} \right|_{t=0^+} = -100V \Rightarrow \left. \frac{di_L(t)}{dt} \right|_{t=0^+} = \frac{-100}{4} = -25 \text{ amp./sec.}$$

Example: L.10.5 Fig. 10.14(a) shows that a switch 'S' has been in position '1' for a long time and is moved in position '2' at time ' $t=0$ '. Find the expression $v(t)$ for $t \geq 0$.

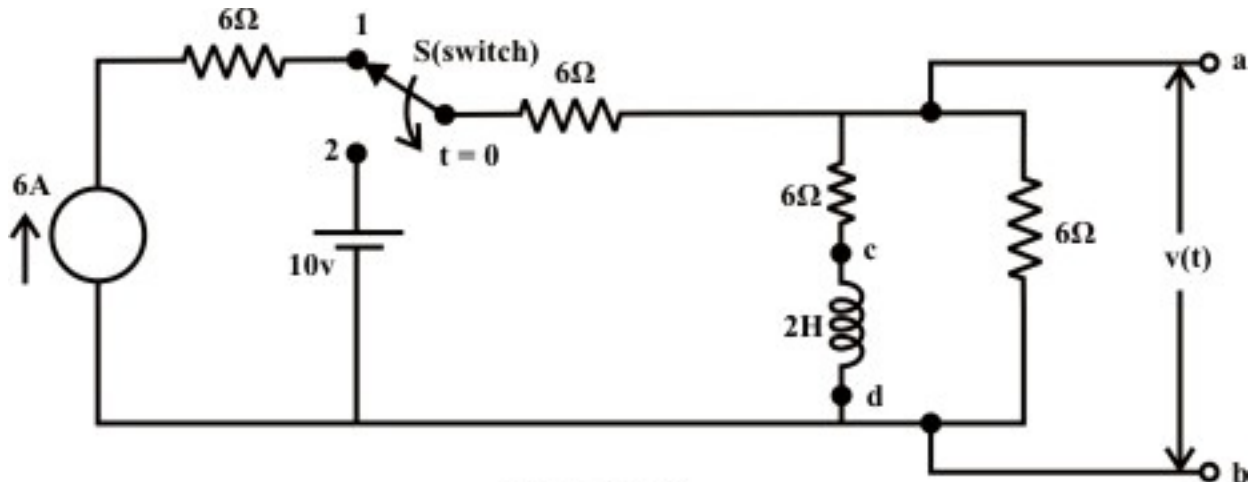


Fig. 10.14(a)

Solution: When the switch 'S' is in position '1', the current through inductor (using the fundamental property of inductor currents) at steady state condition (see fig.10.14(b)) is

$$I_L = \frac{6}{6+6} \times 6 = 3A \Rightarrow I_L(0^-) = I_L(0^+) = 3A \quad (10.31)$$

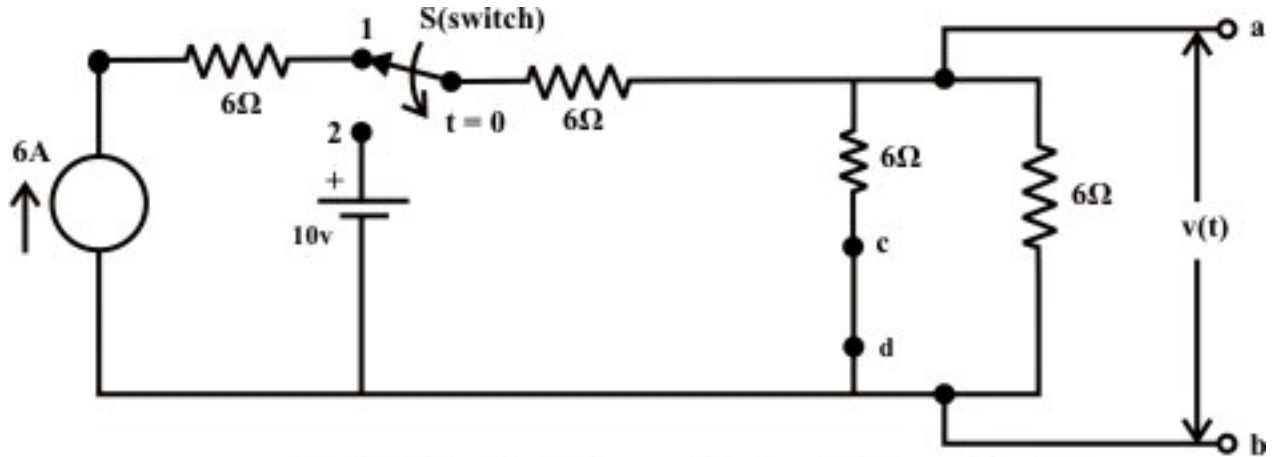


Fig. 10.14(b): Circuit diagram for the switch in position '1'

The circuit for the switch 'S' is in position '2' is shown in fig.10.14 (c). The current in inductor can be computed using following two different methods.

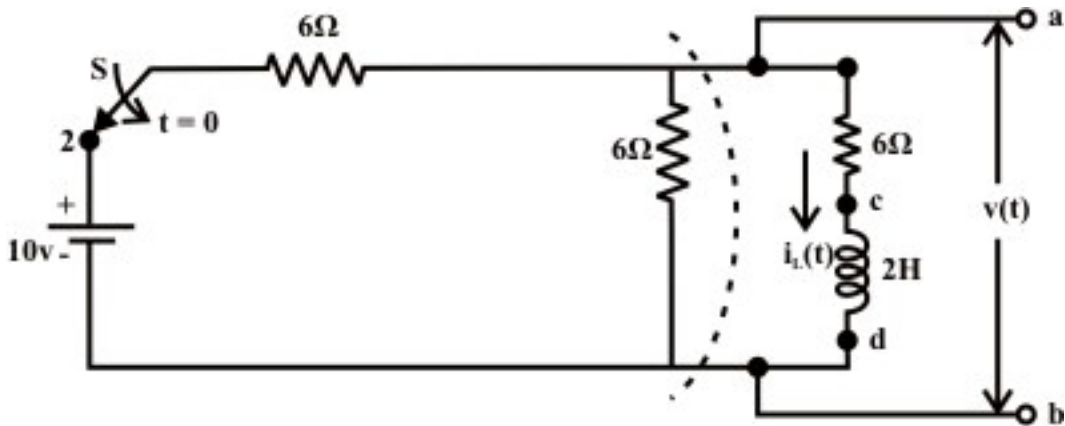


Fig. 10.14(c)

Method-1: Using Thevenin's theorem

Convert the part of a circuit **containing independent sources and resistances** into an equivalent Thevenin's voltage source as shown in fig.10.14.(d).

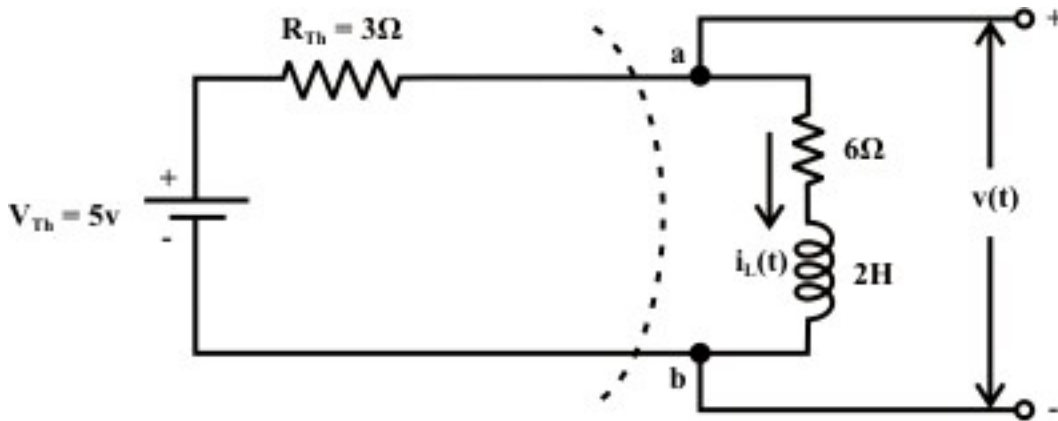


Fig. 10.14(d)

Using the KVL around the closed path is

$$9i_L(t) + 2 \times \frac{di_L(t)}{dt} = 5 \quad (10.32)$$

The solution of the above equation is given by

$$i_L(t) = A_1 e^{-\frac{9}{2}t} + A = i_n(t) + i_f(t) \quad (10.33)$$

where, $i_n(t)$ = complementary/natural/transient solution of eq.(10.32)

$i_f(t)$ = particular/ steady state/final solution of eq.(10.32)

The constants A_1 and A are computed using the initial and final conditions of the circuit when the switch is kept in position '2'.

At time $t = 0$,

$$i_L(0) = i_L(0^+) = 3 = A_1 + A \quad (10.34)$$

At time $t \rightarrow \infty$, the current in inductor reached its steady state condition and acts as a short circuit in a dc source network. The current through inductor is

$$i_L(t = \infty) = \frac{5}{3+6} = 0.555 \text{ amp.} = i_f = A \quad (10.35)$$

Using the above two equations in (10.33), one can obtain the final voltage expression for voltage $v(t)$ across the terminals 'a' and 'b' as

$$v_{ab}(t) = v(t) = 5 - i_L(t) \times 3 = 5 - \left(2.445 \times e^{-\frac{9}{2}t} + 0.555 \right) \times 3 = \left(3.339 - 7.335 \times e^{-\frac{9}{2}t} \right) \text{ V}$$

Method-2: Mesh current method

Assign the loop currents in clockwise directions and redrawn the circuit as shown in Fig. 10.14(e). The voltage across the terminals 'a' and 'b' can be obtained by solving the following loop equations.

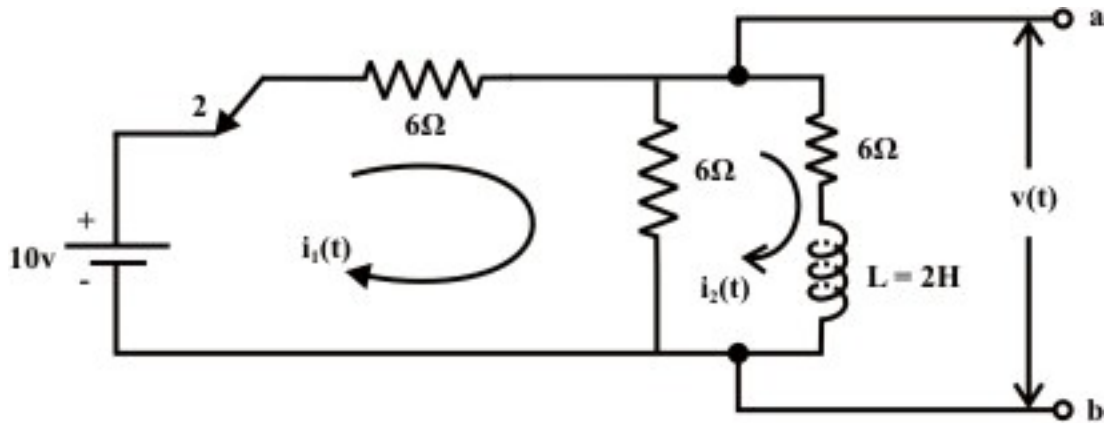


Fig. 10.14(e)

Loop-1:

$$10 - 6i_1(t) - 6(i_1(t) - i_2(t)) = 0 \Rightarrow 10 = 12i_1(t) - 6i_2(t) \Rightarrow i_1(t) = \frac{1}{12}(10 + 6i_2(t)) \quad (10.36)$$

Loop-2:

$$-6i_2(t) - L \frac{di_2(t)}{dt} - 6(i_2(t) - i_1(t)) = 0 \Rightarrow -6i_1(t) + 12i_2(t) + 2 \frac{di_2(t)}{dt} = 0 \quad (10.37)$$

Using the value of $i_1(t)$ in equation (10.37), we get

$$9i_2(t) + 2 \times \frac{di_2(t)}{dt} = 5 \quad (10.38)$$

To solve the above first order differential equation we must know inductor's initial and final conditions and their values are already known (see, $\Rightarrow i_2(0^-) = i_2(0^+) = 3A$ and $i_2(t = \infty) = \frac{5}{3+6} = 0.555 \text{ amp.}$). The solution of differential equation (10.38) provides an expression of current $i_2(t)$ and this, in turn, will give us the expression of $i_1(t)$. The voltage across the terminals 'a' and 'b' is given by

$$v_{ab} = 10 - 6 \times i_1(t) = 6i_2(t) + 2 \frac{di_2(t)}{dt} = \left(3.339 - 7.335 \times e^{-\frac{9}{2}t} \right) V$$

where, $i_2(t)$, $i_1(t)$ can be obtained by solving equations (10.38) and (10.36). The expressions for $i_2(t)$ and hence $i_1(t)$ are given below:

$$i_2(t) = \left(2.445 \times e^{-\frac{9}{2}t} + 0.555 \right) \text{ and } i_1(t) = \frac{1}{12}(10 + 6i_2(t)) = \left(1.11 + 1.2225 e^{-\frac{9}{2}t} \right)$$

L.10.4 Capacitor and its behavior

Fig.10.15 shows a capacitor consists of two pieces of metal (the plates) separated from each other by a good insulator (the dielectric), with two wires (the leads) attached to the metal plates.

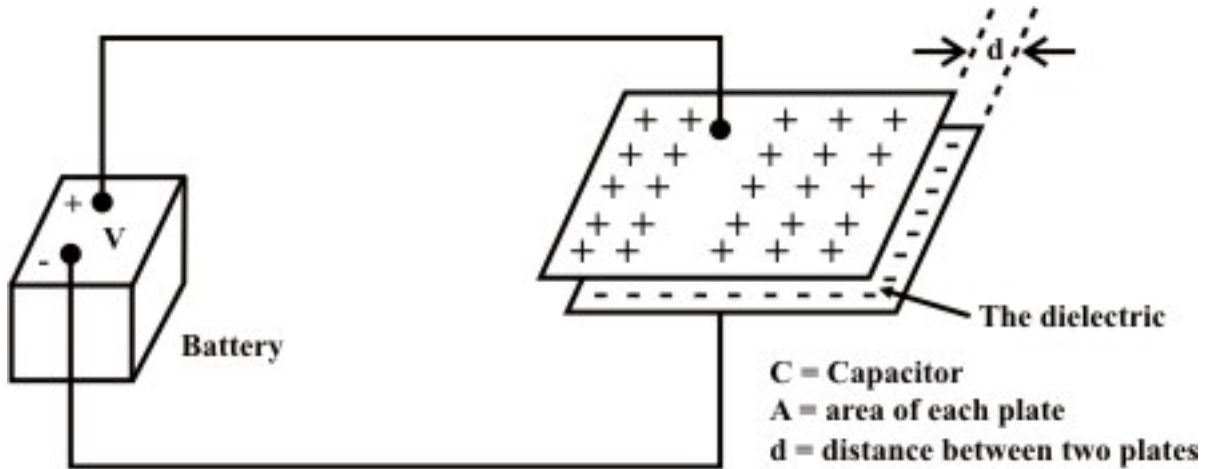


Fig. 10.15: Charging of a Capacitor

A battery is connected across the capacitor to transport charge from one plate to the other until the capacitor charge voltage buildup is equal to the battery voltage V . The voltage across the capacitor depends on how much charge was deposited on the plates and also how much capacitance the capacitor has. In other words, there is a relationship between the voltage (V), charge (Q) and capacitance (C), they are related with a mathematical expression as

$$C = \frac{Q \text{ (coulumb)}}{V \text{ (volt)}} \quad (10.39)$$

where Q = magnitude of charge stored on each plate, V = voltage applied to the plates and the unit of capacitance is in *Farad*. Although the capacitance C of a capacitor is the ratio of charge per plate to the applied voltage but it mainly depends on the physical dimension of the capacitor. If the area of the plates is larger, the more would be the amount of charge stored over the surface of the plates, resulting higher value of capacitance. On the other hand, if the spacing ' d ' between the plates is closer, accumulates more charge over the parallel plates and thus increases the value of the capacitance. The quality of dielectric material has an effect on capacitance between the plates. The good quality of dielectric material indicates that higher the permittivity, resulting greater the capacitance. The value of capacitance can be expressed in terms physical parameters of capacitor as

$C = \frac{\epsilon A}{d} = \frac{\epsilon_0 \epsilon_r A}{d}$ where A is the area of each plate, d is the distance between the plates, ϵ_0 ($= 8.85 \times 10^{-12}$) is the permittivity of free-space, ϵ_r = relative permittivity of dielectric material and C is the capacitance in Farad. It is important to note that when the applied

voltage across the capacitor exceeds a certain value the dielectric material breaks down and loses its insulation property.

L.10.4.1 Continuity condition of capacitors

To find the current-voltage relationship of the capacitor, one can take the derivative of both sides of Eq.(10.39)

$$C \frac{d v_c(t)}{dt} = \frac{d q(t)}{dt} = i(t) \Rightarrow i(t) = C \frac{d v_c(t)}{dt} \quad (10.40)$$

The voltage-current relation can also be represented by another form as

$$v_c(t) = \frac{1}{C} \int_{t_0}^t i(t) dt + v_c(t_0) \quad \text{where } v_c(t_0) \text{ is voltage across the capacitor at time ' } t_0 \text{ ' . It can}$$

be seen that when the voltage across a capacitor is not changing with time, or, in other words, the capacitor is fully charged and the current through the capacitor is zero (see Eq.10.40). This means that the capacitor resembles as an open circuit and blocks the flow of current through the capacitor. Equation (10.40) shows that an instantaneous ($\Delta t=0$) change in capacitance voltage must be accompanied by an infinite current that requiring an infinite power source. In practice, this situation will not occur in any circuits containing energy storing elements. Thus, the voltage across the capacitor (or electric charge $q(t)$) cannot change instantaneously (i.e., $\Delta t=0$), that is we cannot have any discontinuity in voltage across the capacitor.

Remark-5

(i) The voltage across and charge on a capacitor cannot change instantaneously (i.e. $v_c(0^-)$ just right before the change of voltage = $v_c(0^+)$ just right after the change of voltage). However, current through a capacitor can change abruptly. (ii) The capacitor acts as an open circuit (i.e., when the capacitor is fully charged) when voltage across the capacitor does not change (constant). (iii) These properties of capacitor are important since they will be used to determine “boundary conditions”.

L.10.4.2 Study of dc transients and steady state response of a series R-C circuit.

Ideal and real capacitors: An ideal capacitor has an infinite dielectric resistance and plates (made of metals) that have zero resistance. However, an ideal capacitor does not exist as all dielectrics have some leakage current and all capacitor plates have some resistance. A capacitor’s leakage resistance is a measure of how much charge (current) it will allow to leak through the dielectric medium. Ideally, a charged capacitor is not supposed to allow leaking any current through the dielectric medium and also assumed not to dissipate any power loss in capacitor plates resistance. Under this situation, the model as shown in fig. 10.16(a) represents the ideal capacitor. However, all real or practical capacitor leaks current to some extent due to leakage resistance of dielectric medium. This leakage resistance can be visualized as a resistance connected in parallel

with the capacitor and power loss in capacitor plates can be realized with a resistance connected in series with capacitor. The model of a real capacitor is shown in fig. 10.16(b).

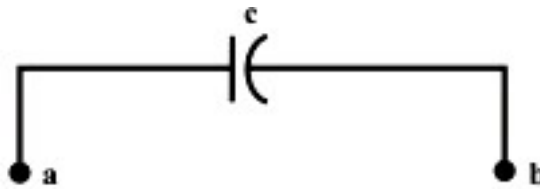


Fig. 10.16(a): Symbolic representation of an ideal capacitor

In present discussion, an ideal capacitor is considered to study the behavior of dc transients in $R-C$ circuit.

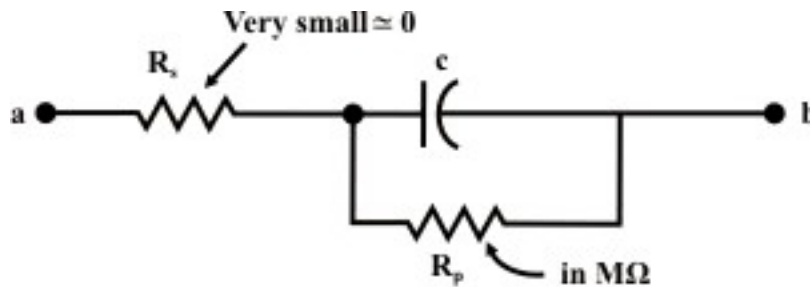


Fig. 10.16(b): Symbolic representation of a real capacitor

L.10.4.3 Charging of a capacitor or Growth of a capacitor voltage in dc circuits

Let us consider a simple series $R-C$ circuit shown in fig. 10.17(a) is connected through a switch 'S' to a constant voltage source V_s .

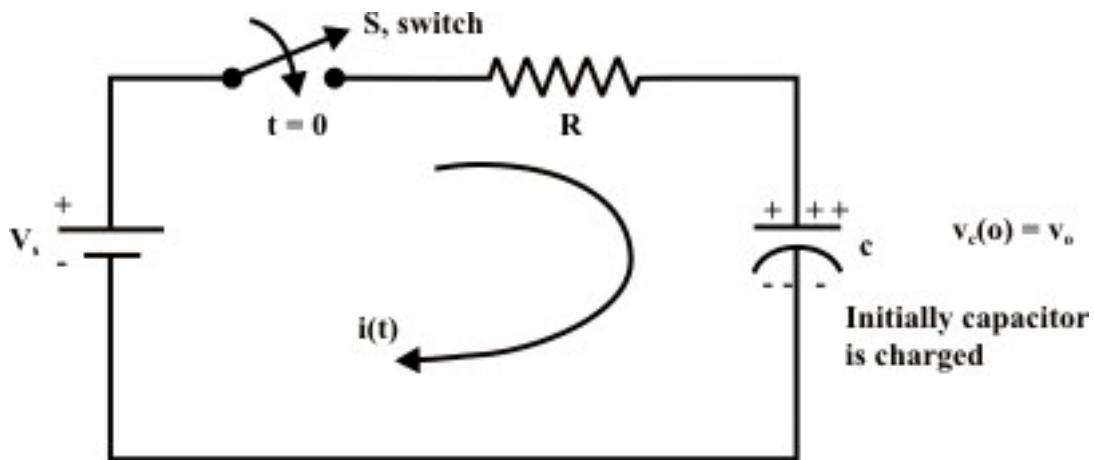


Fig. 10.17(a): Charging of a RC circuit

The switch 'S' is closed at time 't=0' (see fig. 10.7(a)). It is assumed that the capacitor is initially charged with a voltage $v_c(0) = v_0$ and the current flowing through the circuit at any instant of time 't' after closing the switch is $i(t)$.

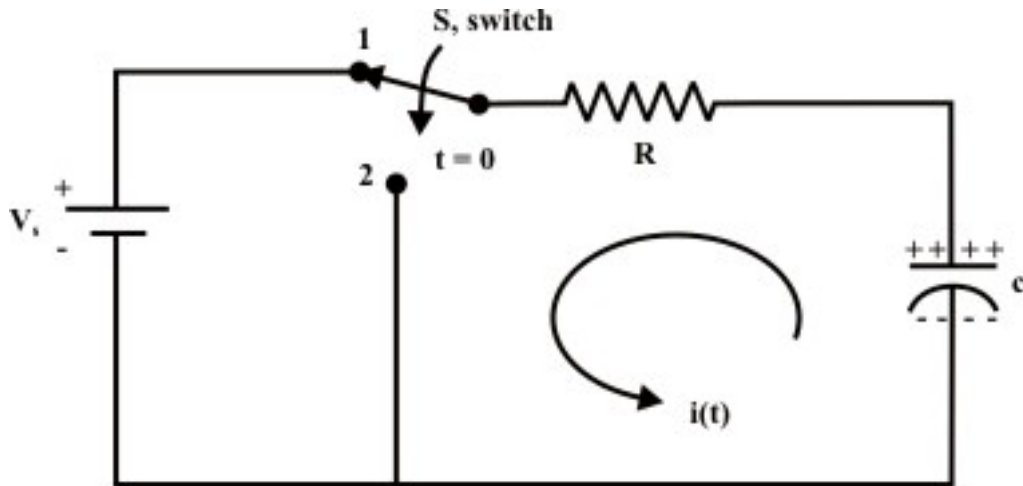


Fig. 10.17(b): Discharging of a RC circuit

The KVL equation around the loop can be written as

$$V_s = R i(t) + v_c(t) \Rightarrow V_s = R C \frac{d v_c(t)}{dt} + v_c(t) \quad (10.41)$$

The solution of the above first-order differential equation (10.41) due to forcing function V_s is given by

$$v_c(t) = v_{cn}(t) \text{ (natural response/transient response)} + v_{cf}(t) \text{ (steady-state response)}$$

$$= A_1 e^{\alpha t} + A \quad (10.42)$$

The constants A_1 , and A are computed using the initial and boundary conditions. The value of α is obtained from the characteristic equation given by (see in detail in Appendix)

$$R C \alpha + 1 = 0 \Rightarrow \alpha = -\frac{1}{R C}$$

Eq. (10.42) is then rewritten as

$$v_c(t) = A_1 e^{-\frac{1}{R C} t} + A \quad (10.43)$$

At steady state, the voltage across the capacitor is $v_c(\infty) = v_{cf} = A$ which satisfy the original differential equation (10.41). i.e.,

$$V_s = R C \frac{d v_{cf}}{dt} + v_{cf} \Rightarrow R C \frac{d A}{dt} + A \Rightarrow A = V_s$$

Using the initial condition (at $t=0$) in equation (10.43), we get

$$v_c(0) = v_0 = A_1 e^{-\frac{1}{RC} \times 0} + A \Rightarrow A_1 = v_0 - A = v_0 - V_S$$

The values of A_1 , A , and Eq. (10.43) together will give us the final expression for capacitor voltage as

$$v_c(t) = (v_0 - V_S) e^{-\frac{1}{RC} t} + V_S \Rightarrow v_c(t) = V_S \left(1 - e^{-\frac{1}{RC} t} \right) + v_0 e^{-\frac{1}{RC} t} \quad (10.44)$$

Thus,

$$v_c(t) = \begin{cases} v_0 & t < 0 \\ v_c(t) = V_S \left(1 - e^{-\frac{1}{RC} t} \right) + v_0 e^{-\frac{1}{RC} t} & t > 0 \end{cases}$$

Response of capacitor voltage with time is shown in fig. 10.18.

Special Case: Assume initial voltage across the capacitor at time ' $t=0$ ' is zero i.e., $v_c(0) = v_0 = 0$. The voltage expression for capacitor at any instant of time can be written from Eq.(10.44) with $v_c(0) = v_0 = 0$.

$$\text{Voltage across the capacitance } v_c(t) = V_S \left(1 - e^{-\frac{1}{RC} t} \right) \quad (10.45)$$

$$\text{Voltage across the resistance } v_R(t) = V_S - v_c(t) = V_S e^{-\frac{1}{RC} t} \quad (10.46)$$

$$\text{Charging current through the capacitor } i(t) = \frac{v_R}{R} = \frac{V_S}{R} e^{-\frac{1}{RC} t} \quad (10.47)$$

Charge accumulated on either plate of capacitor at any instant of time is given by

$$q(t) = C v_c(t) = C V_S \left(1 - e^{-\frac{1}{RC} t} \right) = Q \left(1 - e^{-\frac{1}{RC} t} \right) \quad (10.48)$$

where Q is the final charge accumulated in the capacitor at steady state (i.e., $t \rightarrow \infty$). Once the voltage across the capacitor $v_c(t)$ is known, the other quantities (like, $v_R(t)$, $i(t)$, and $q(t)$) can easily be computed using the above expressions. Fig. 10.19(a) shows growth of capacitor voltage $v_c(t)$ for different choices of circuit parameters (assumed that the capacitor is initially not charged). A sketch for $q(t)$ and $i(t)$ is shown in fig. 10.19(b).

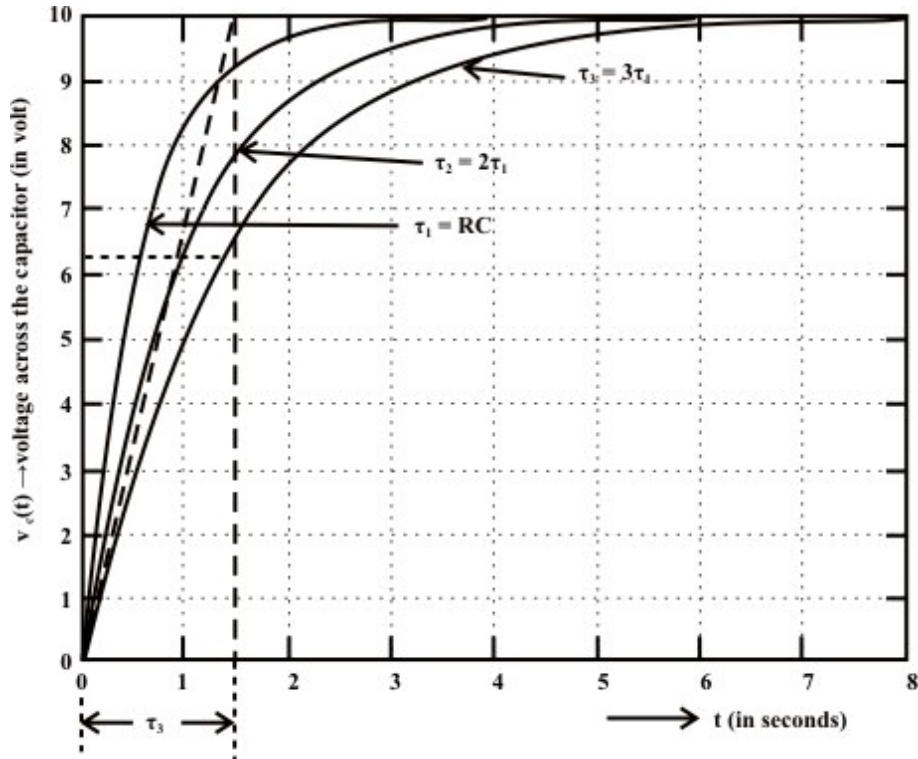


Fig. 10.19(a): Growth of capacitor voltage (assumed initial capacitor voltage is zero)

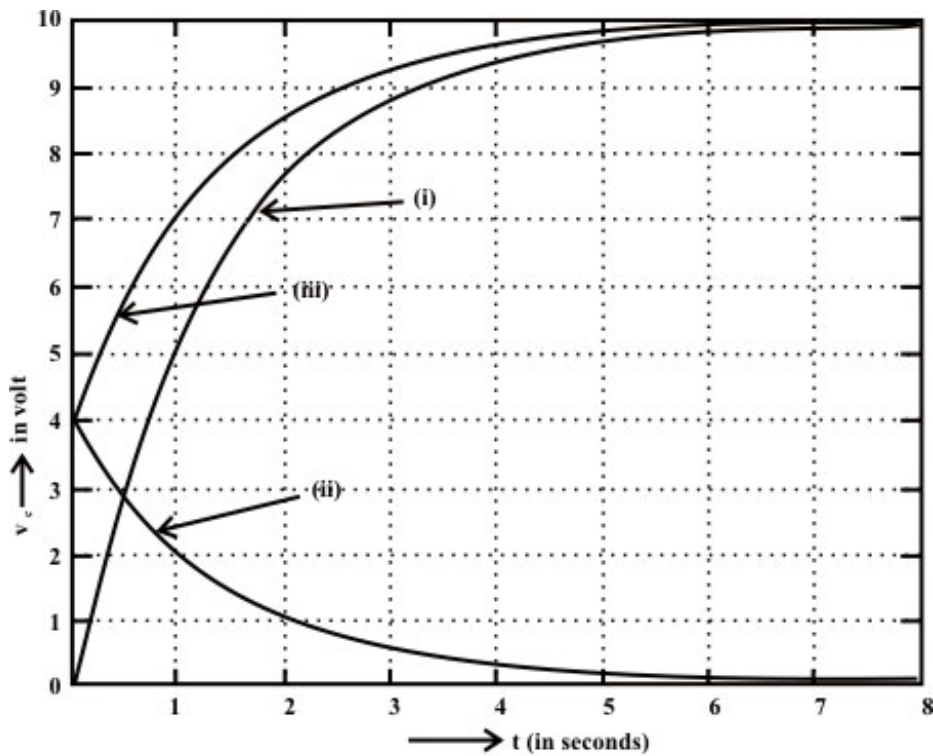


Fig. 10.18: Voltage across the capacitor due to
 (i) the forcing function V_s acting alone
 (ii) discharge of capacitor initial voltage v_0
 (iii) Combine effect of (i) and (ii)

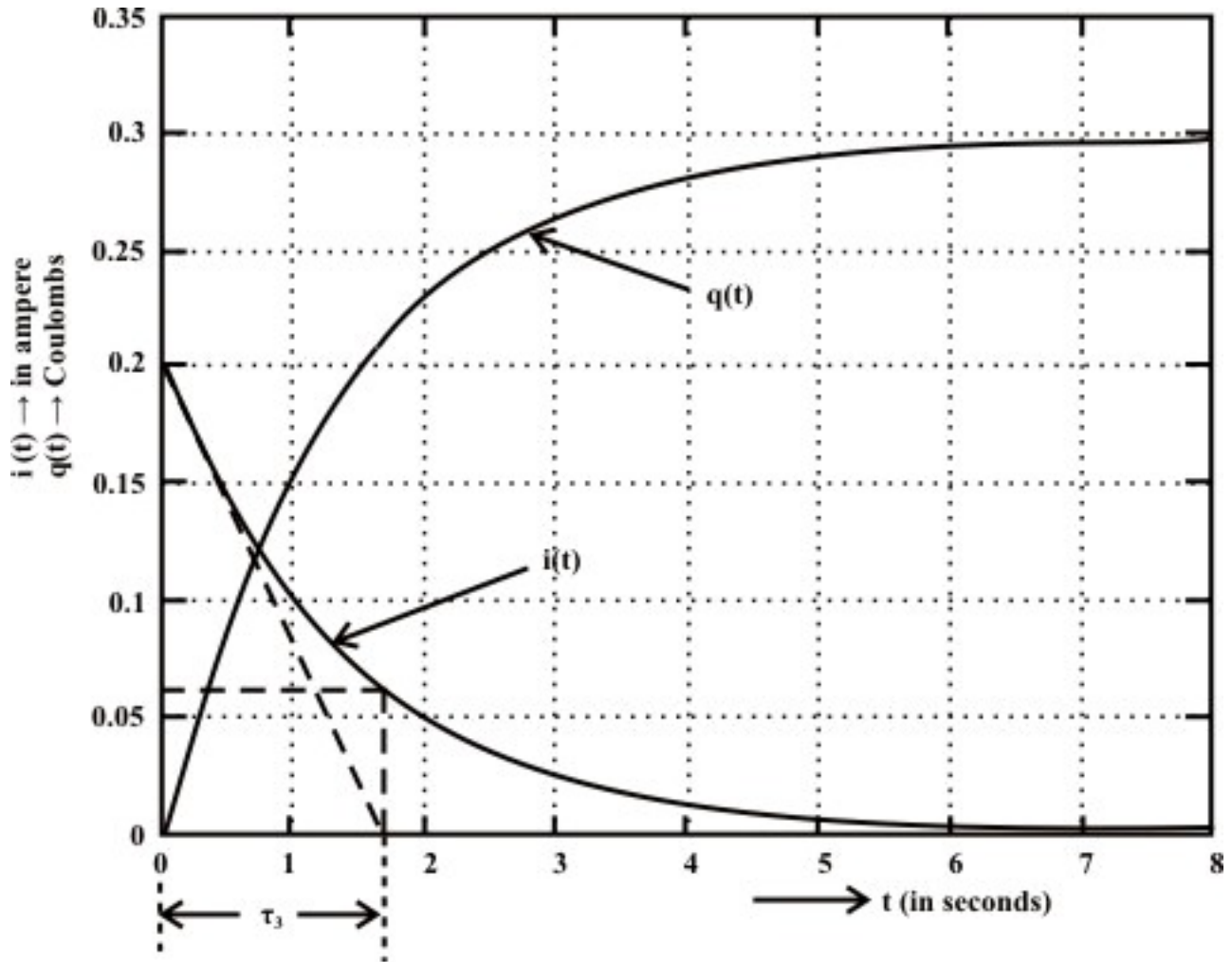


Fig. 10.19(b): System response due to the forcing function V_s (assumed capacitor initial voltage $v_o = 0$)

Following the definition given in section L.10.3.1, time constant of each of the exponential expressions described in Eqs. 10.45 to 10.48 may be found as $\tau = RC$ (for RC circuit).

L.10.4.4 Discharging of a capacitor or Fall of a capacitor voltage in dc circuits

Fig. 10.17(b) shows that the switch 'S' is closed at position '1' for sufficiently long time and the circuit has reached in steady-state condition. At ' $t=0$ ' the switch 'S' is opened and kept in position '2' and remains there. Our job is to find the expressions for (i) voltage across the capacitor (v_c) (ii) voltage across the resistance (v_R) (iii) current ($i(t)$) through the capacitor (discharging current) (iv) discharge of charge ($q(t)$) through the circuit.

Solution: For $t < 0$, the switch 'S' in position 1. The capacitor acts like an open circuit to dc, but the voltage across the capacitor is same as the supply voltage V_s . Since, the capacitor voltage cannot change instantaneously, this implies that

$$v_c(0^-) = v_c(0^+) = V_s$$

When the switch is closed in position '2', the current $i(t)$ will flow through the circuit until capacitor is completely discharged through the resistance R . In other words, the discharging cycle will start at $t = 0$. Now applying KVL around the loop, we get

$$R C \frac{dv_c(t)}{dt} + v_c(t) = 0 \quad (10.49)$$

The solution of input free differential equation (10.49) is given by

$$v_c(t) = A_1 e^{\alpha t} \quad (10.50)$$

where the value of α is obtained from the characteristic equation and it is equal to $\alpha = -\frac{1}{RC}$. The constant A_1 is obtained using the initial condition of the circuit in Eq.(10.50). Note, at ' $t=0$ ' (when the switch is just closed in position '2') the voltage across the capacitor $v_c(t) = V_s$. Using this condition in Eq.(10.50), we get

Using this condition in Eq.(10.50), we get

$$v_c(0) = V_s = A_1 e^{-\frac{1}{RC} \times 0} \Rightarrow A_1 = V_s$$

Now the following expressions are written as

$$\text{Voltage across the capacitance } v_c(t) = V_s e^{-\frac{1}{RC} t} \quad (10.51)$$

$$\text{Voltage across the resistance } v_R(t) = -v_c(t) = -V_s e^{-\frac{1}{RC} t} \quad (10.52)$$

$$\text{Charging current through the capacitor } i(t) = \frac{v_R}{R} = -\frac{V_s}{R} e^{-\frac{1}{RC} t} \quad (10.53)$$

An inspection of the above exponential terms of equations from (10.51) to (10.53) reveals that the time constant of RC circuit is given by

$$\tau = RC \text{ (sec.)}$$

This means that at time $t = \tau$, the capacitor's voltage v_c drops to 36.8% of its initial value (see fig. 10.20(a)). For all practical purposes, the dc transient is considered to end after a time span of 5τ . At such time steady state condition is said to be reached. Plots of above equations as a function of time are depicted in fig. 10.20(a) and fig. 10.20(b) respectively.

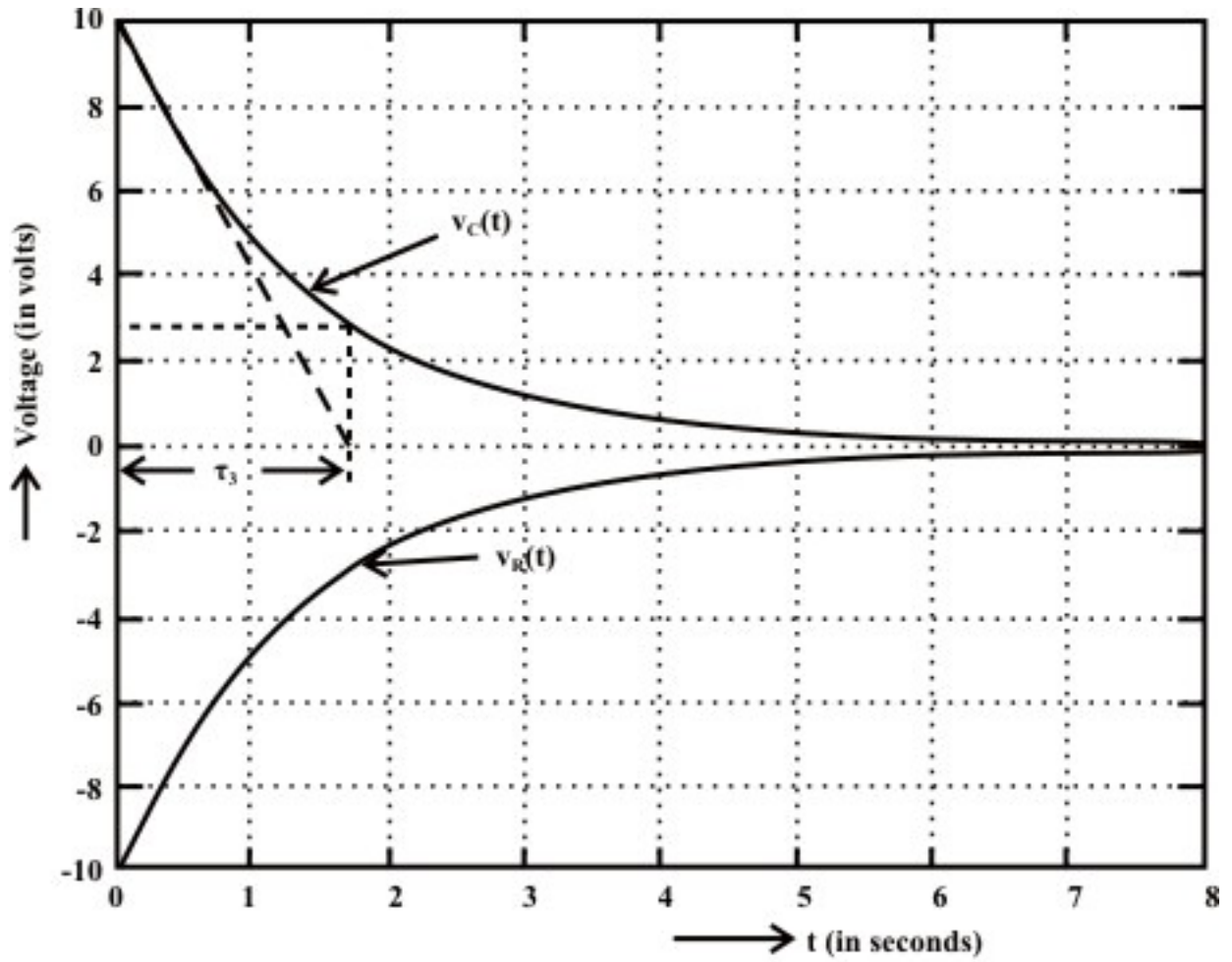


Fig. 10.20(a): Discharge of capacitor voltage with time in R-C circuit

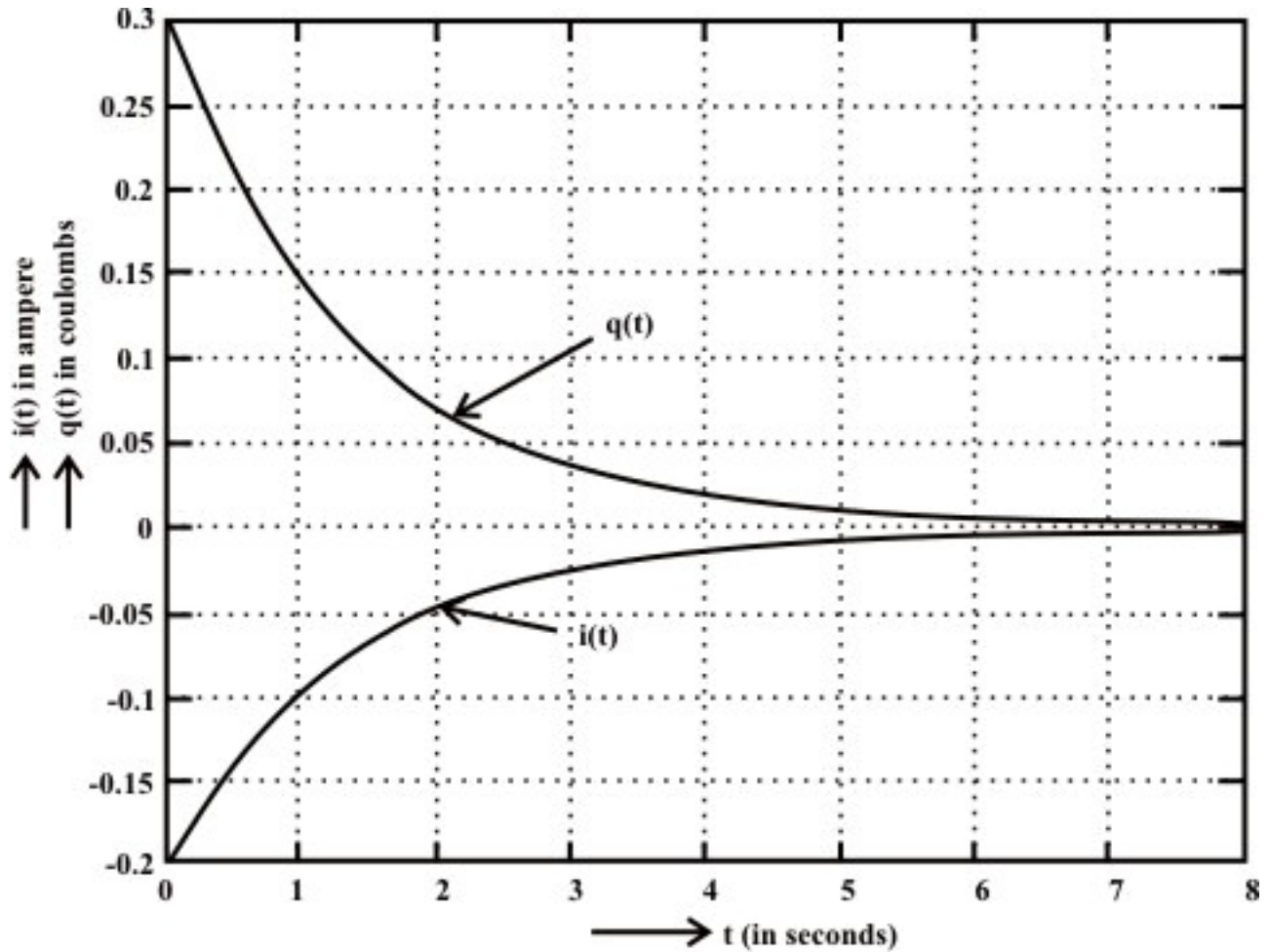


Fig. 10.20(b): System response due to capacitor discharge

L.10.5 Energy stored in a capacitor

The ideal capacitor does not dissipate any of the energy supplied by the source. It stores energy in the form of an electric field between the conducting plates. Let us consider a voltage source V_s is connected to a series $R-C$ circuit and it is assumed that the capacitor is initially uncharged. The capacitor voltage ($v_c(t)$) and current ($i_c(t)$) waveforms during the charging period are shown in fig.10.21 (see the expressions (10.45) and (10.47)) and instantaneous power ($p_c(t) = v_c(t) \times i(t)$) supplied to the capacitor is also shown in the same figure.

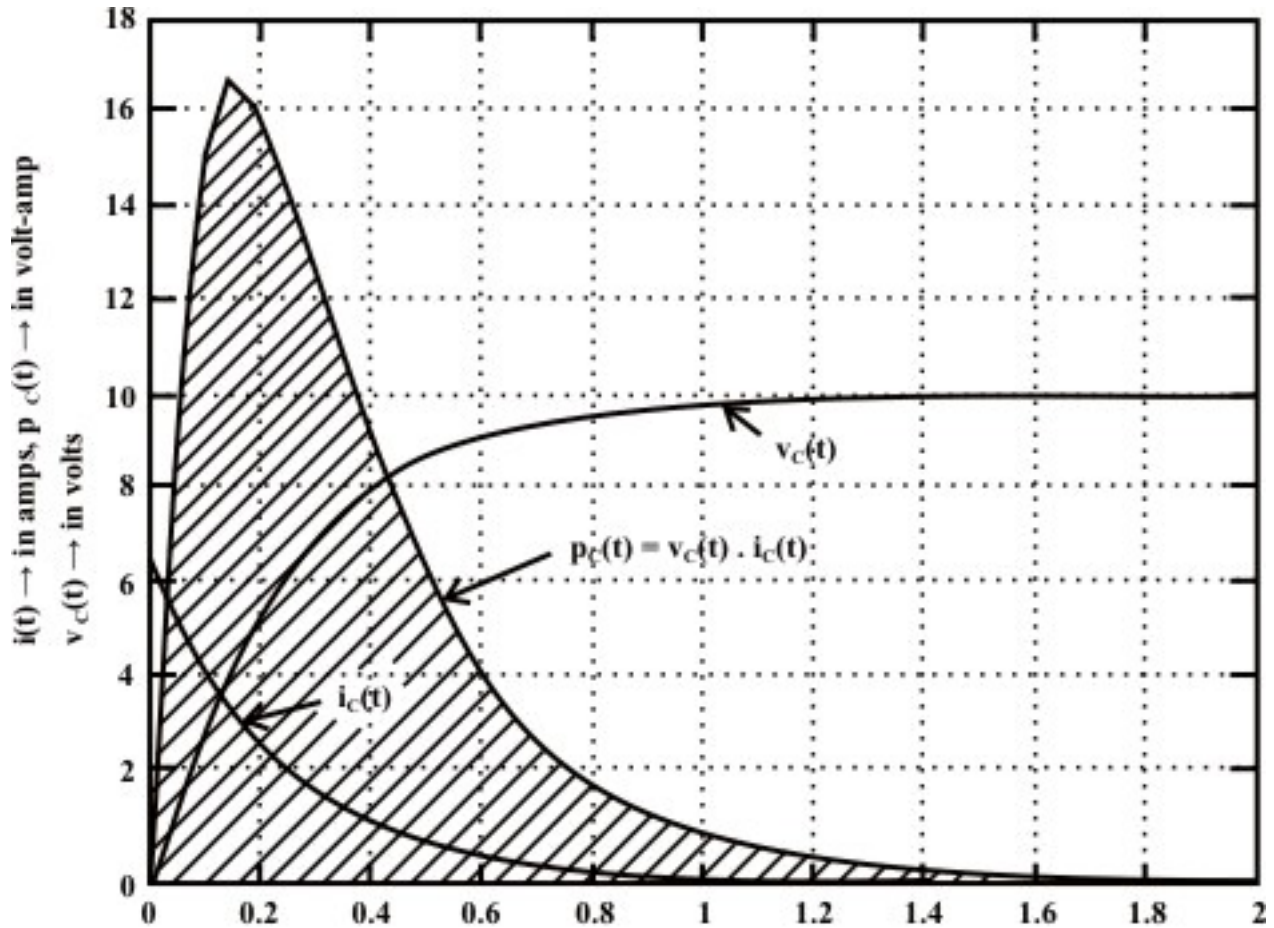


Fig. 10.21: Instantaneous power delivered to the capacitor

Let us consider the instantaneous power supplied to the capacitor is given by

$$p_c(t) = v_c(t) \times i(t) \quad (10.54)$$

Now, the energy supplied to the capacitor in dt second is given by

$$\Delta w = p_c(t) \times dt = v_c(t) C \frac{dv_c(t)}{dt} \times dt = C v_c(t) dv_c(t) \quad (10.55)$$

Total energy supplied to the capacitor in t seconds is expressed as

$$w(t) = C \int_{v_c(0)=0}^{v_c(t)=v} v_c(t) dv_c(t) = \frac{1}{2} C v_c^2 = \frac{1}{2} \frac{q^2(t)}{C} (\text{Joules}) \quad (10.56)$$

(Note initial voltage across capacitor is zero and $q(t)$ is the charge accumulated on each plate at a time t).

When the capacitor is fully charged, its terminal voltage is equal to the source voltage V_s .

The amount of energy stored in capacitor in the form of electric field is given by

$$W = \frac{1}{2} C V_s^2 = \frac{1}{2} \frac{Q^2}{C} (\text{Joules}) \quad (10.57)$$

where Q is the final charge accumulated on each plate of the capacitor at steady state (i.e., $t \rightarrow \infty$) i.e., when the capacitor is fully charged.

Example: L.10.6 The switch ‘ S ’ shown in fig.L.10.22 is kept open for a long time and then it is closed at time ‘ $t=0$ ’. Find (i) $v_c(0^-)$ (ii) $v_c(0^+)$ (iii) $i_c(0^-)$ (iv) $i_c(0^+)$ (v)

$\left. \frac{dv_c(t)}{dt} \right|_{t=0^+}$ (vi) find the time constants of the circuit before and after the switch is closed

(iv) $v_c(\infty)$

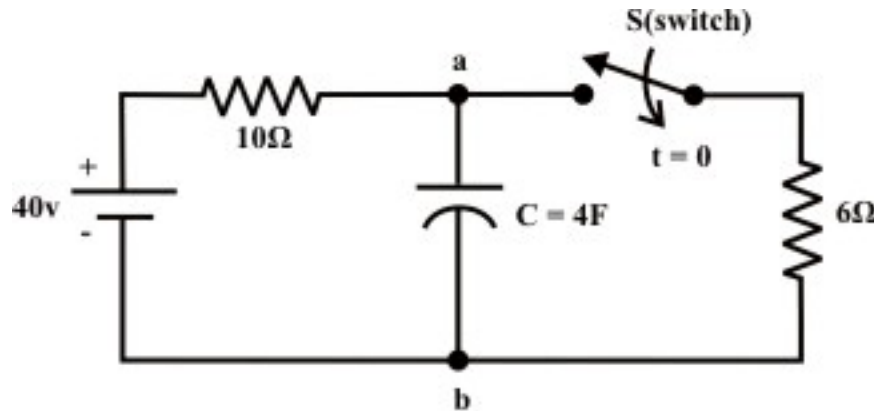


Fig. 10.22

Solution: As we know the voltage across the capacitor $v_c(t)$ cannot change instantaneously due to the principle of conservation of charge. Therefore, the voltage across the capacitor just before the switch is closed $v_c(0^-) =$ voltage across the capacitor just after the switch is closed $v_c(0^+) = 40\text{ V}$ (note the terminal ‘ a ’ is positively charged. It may be noted that the capacitor current before the switch ‘ S ’ is closed is $i_c(0^-) = 0\text{ A}$. On the other hand, at $t=0$, the current through 10Ω resistor is zero but the current through capacitor can be computed as

$i_c(0^+) = \frac{v_c(0)}{6} = \frac{40}{6} = 6.66\text{ A}$ (note, voltage across the capacitor cannot change instantaneously at instant of switching). The rate of change of capacitor voltage at time ‘ $t=0$ ’ is expressed as

$$C \left. \frac{dv_c(t)}{dt} \right|_{t=0^+} = i_c(0) \Rightarrow \frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{6.66}{4} = 1.665\text{ volt/sec.}$$

Time constant of the circuit before the switch was closed = $\tau = RC = 10 \times 4 = 40\text{ sec}$. Time constant of the circuit after the switch is closed is $\tau = R_{th} C = \frac{10 \times 6}{10 + 6} \times 4 = 15\text{ sec}$. (replace the part of the circuit than contains only independent sources and resistive elements by an

equivalent Thevenin's voltage source. In this case, we need only to find the Thevenin resistance R_{Th}).

Note: When the switch is kept in closed position, initially the capacitor will be in discharge state and subsequently its voltage will decrease with the increase in time. Finally, at steady state the capacitor is charged with a voltage $v_c(t=\infty) = \frac{40}{10+6} \times 6 = 15\text{ V}$ (theoretically, time required to reach the capacitor voltage at steady value is $5\tau = 5 \times 15 = 75\text{ sec.}$).

Example: L.10.7 The circuit shown in Fig.10.23 has been established for a long time. The switch is closed at time $t=0$. Find the current (i) $i_1(0^+)$, $i_2(0^+)$, $i_3(0^+)$, and $\left. \frac{dv_{de}}{dt} \right|_{t=0^+}$ (ii) at steady state the voltage across the capacitors, $i_1(\infty)$, $i_2(\infty)$ and $i_3(\infty)$.

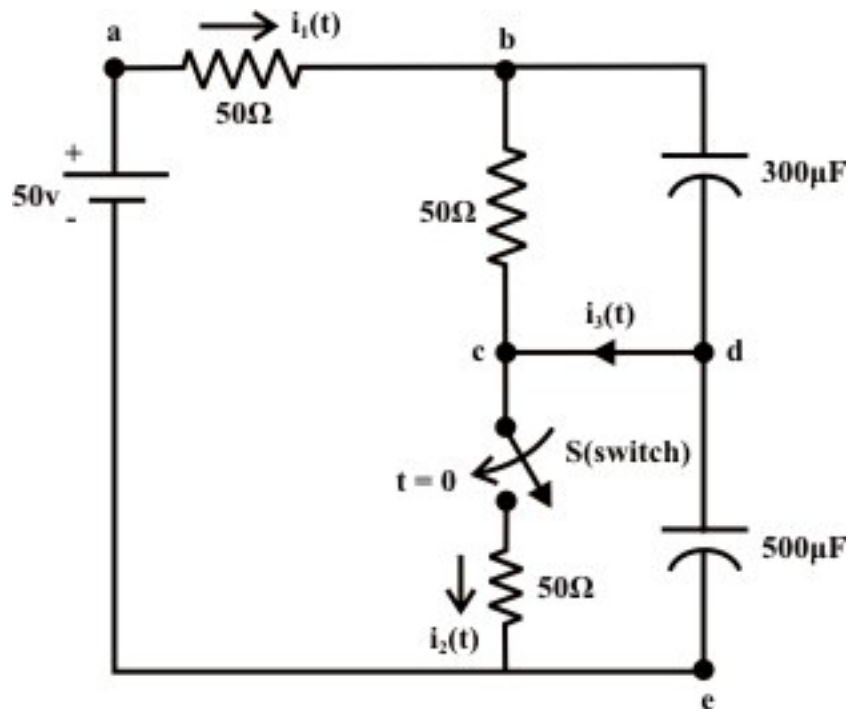


Fig. 10.23

Solution: (i) At $t=0^-$ no current flowing through the circuit, so the voltage at points 'b' and 'd' are both equal to 50 volt. When the switch 'S' closes the capacitor voltage remains constant and does not change its voltage instantaneously. The current $i_1(0^+)$ through $a-b$ branch must then equal to zero, since voltage at terminal 'b' is equal to $v_b(0^+) = 50\text{ volt.}$, current through $b-c$ is also zero. One can immediately find

out the current through $c-e$ equal to $i_2(0^+) = \frac{50}{50} = 1 \text{ A}$. Applying KCL at point 'c', $i_3(0^+) = 1 \text{ A}$ which is the only current flow at $t=0^+$ around the loop 'd-c-e-d'. Note the capacitor across 'd-e' branch acts as a voltage source, the change of capacitor voltage $\left. \frac{dv_{de}}{dt} \right|_{t=0^+} = \frac{1}{500 \times 10^{-6}} i_3(0^+) = 2 \text{ k volt / sec}$.

(ii) at steady state the voltage across each capacitor is given

$$= \frac{50}{150} \times 50 = 16.666 \text{ volt.}$$

At steady state current delivered by the source to the different branches are given by

$$i_1(\infty) = \frac{50}{150} = 0.333 \text{ A}; \quad i_2(\infty) = 0.333 \text{ A} \quad \text{and} \quad i_3(\infty) = 0 \text{ A}$$

Example: L.10.8 The circuit shown in fig. 10.24(a) is switched on at time $t=0$. How long it takes for the capacitor to attain 70 % of its final voltage? Assume the capacitor is initially not charged. Find also the time constant (τ) of the circuit after the switch is closed.

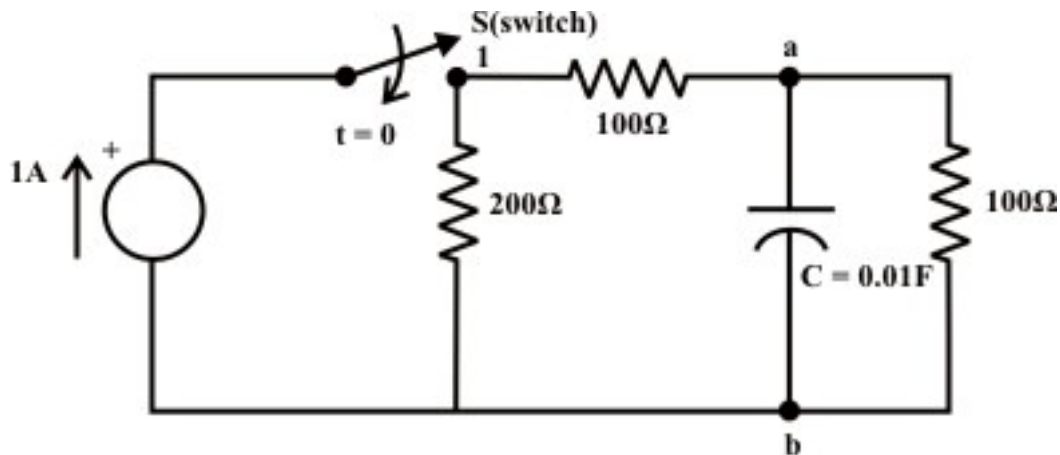


Fig. 10.24(a)

The circuit containing only resistive elements and independent current source (i.e., non-transient part of the circuit) is converted to an equivalent voltage source which is shown in fig.10.24(b).

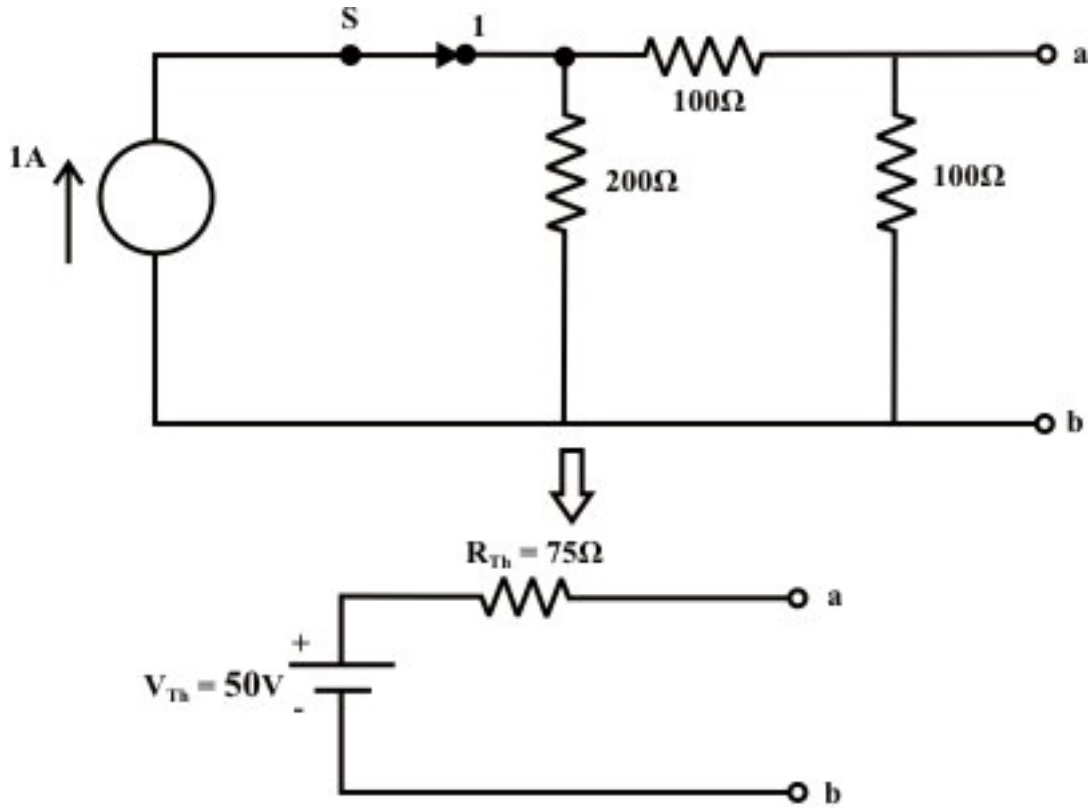


Fig. 10.24(b)

Fig.10.24(c) shows the capacitor C is connected across the Thevenin's voltage terminals 'a' and 'b' in series with Thevenin's resistance R_{Th} .

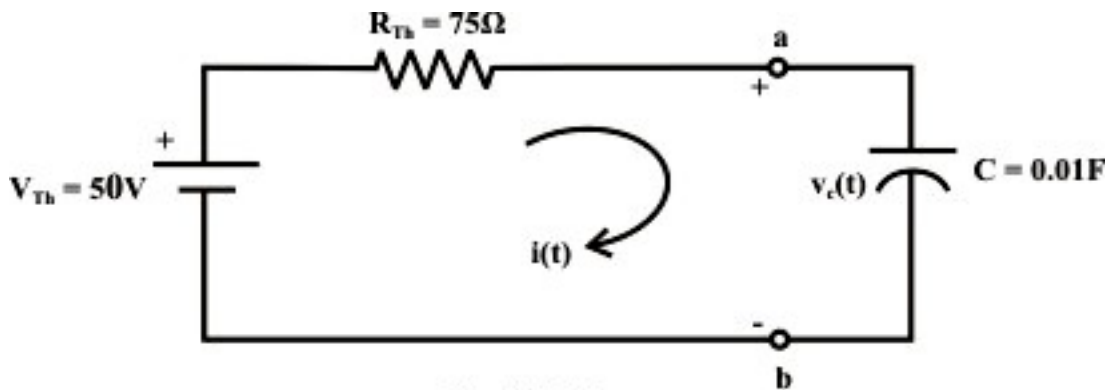


Fig. 10.24(c)

The parameters of Thevenin's voltage source are computed below:

$$V_{Th} = \frac{200}{200+100+100} \times 1 \times 100 = 50V \quad \text{and} \quad R_{Th} = \frac{100 \times 300}{100+300} = 75\Omega$$

Using KVL around the closed path, one can find the current through the capacitor and hence, the voltage across the capacitor.

$$50 = 75 \times i(t) + v_c(t) = 0.75 \frac{dv_c(t)}{dt} + v_c(t) \quad (10.58)$$

The solution of the differential equation is given by

$$v_c(t) = A_1 e^{-\frac{1}{RC}t} + A \quad (10.59)$$

Using the initial and boundary conditions of the circuit, we obtain the final expression of voltage across the capacitor $v_c(t)$ as

$$v_c(t) = 50(1 - e^{-1.33t}) \quad (10.60)$$

Let 't' is the time required to reach the capacitor voltage 70% of its final (i.e., steady state) voltage.

$$50 \times 0.7 = 35 = 50(1 - e^{-1.33t}) \Rightarrow t = 0.91 \text{ sec.}$$

Example: L.10.9 The switch 'S' of the circuit shown in fig.10.25(a) is closed at position '1' at $t=0$.

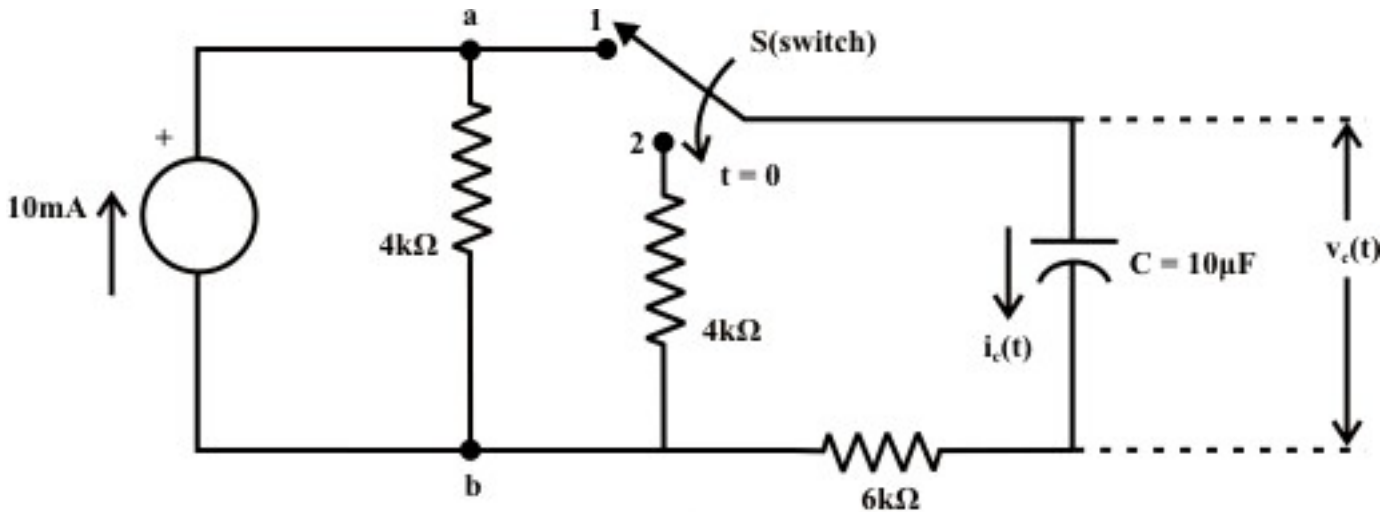


Fig. 10.25(a)

Find voltage $v_c(t)$ and current $i_c(t)$ expressions for $t \geq 0$. Assume that the capacitor is initially fully uncharged (i.e., $v_c(0)=0$).

- (i) find the mathematical expressions for $v_c(t)$ and $i_c(t)$ if the switch 'S' is thrown into position '2' at $t = \tau$ (sec.) of the charging phase.
- (ii) plot the waveforms obtained in parts (i) to (ii) on the same time axis for the voltage $v_c(t)$ and the current $i_c(t)$ using the identified polarity of voltage and current directions.

Solution: (i) The current source is converted to an equivalent voltage source and it is redrawn in fig.10.25(b) when the switch 'S' is in position '1'.

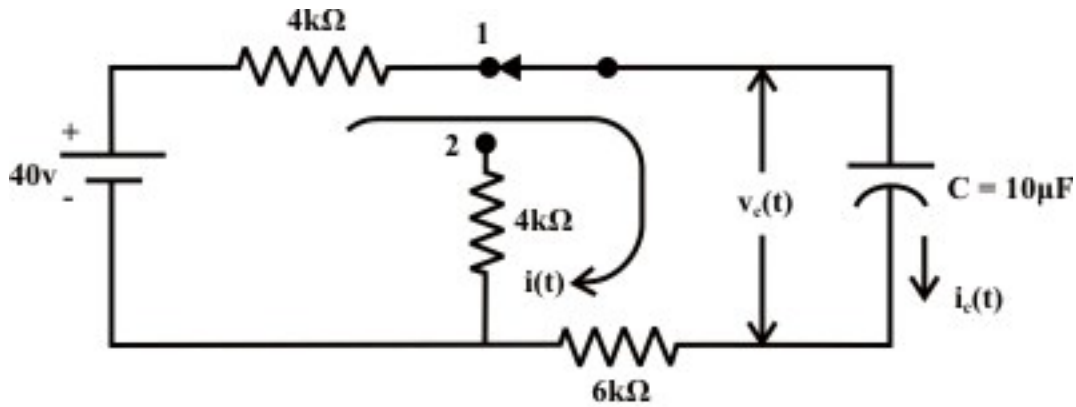


Fig. 10.25(b)

KVL around the closed path:

$$40 = 10 \times i(t) + v_c(t), \quad \text{where } i(t) \text{ is in mA.}$$

$$40 = 10 \times C \frac{dv_c(t)}{dt} + v_c(t) \quad (10.61)$$

The voltage expression across the capacitor using the initial and boundary conditions of the circuit, one can write $v_c(t)$ as

$$v_c(t) = 40 \left(1 - e^{-\frac{t}{RC}} \right) = 40 \left(1 - e^{-\frac{1}{10 \times 10^3 \times 10 \times 10^{-6}} t} \right) = 40 \left(1 - e^{-10t} \right) \quad (10.62)$$

$$i_c(t) = \frac{40 - v_c(t)}{10} = \frac{40 e^{-10t}}{10} = 4 \times e^{-10t} \text{ (in mA)} \quad (10.63)$$

Note that the time constant of the circuit in part (i) is $\tau = RC = 100 \text{ msec.}$

(ii) The switch 'S' is thrown into position '2' at $t = \tau = 0.1 \text{ sec.}$ and the corresponding circuit diagram is shown in fig.10.25(c).

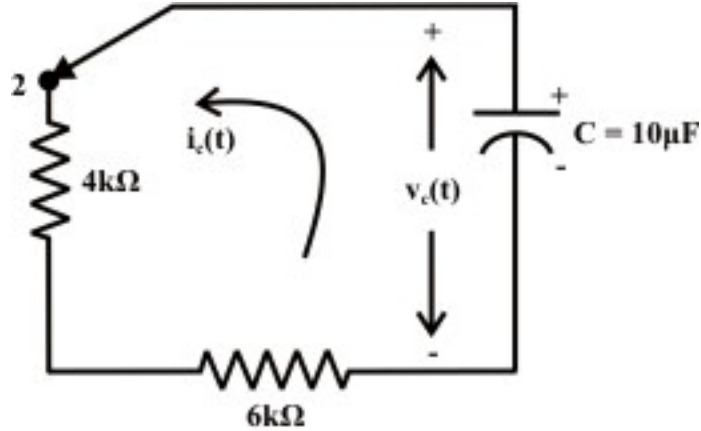


Fig. 10.25(c)

Note, at time $t = \tau = 0.1 \text{ sec.}$, the capacitor is charged with a voltage $= v_c(\tau = 0.1) = 40(1 - e^{-10 \times 0.1}) = 40 \times 0.632 = 25.28 \text{ V}$ and at the same time ($t = \tau = 0.1 \text{ sec.}$) the current in capacitor is $4 \times e^{-10t} = 4 \times 0.368 = 1.472 \text{ (in mA)}$. Considering the fig.10.25(c), one can write KVL around the closed path

$$v_c(t) + C \frac{dv_c(t)}{dt} \times R_{eq} = 0 \tag{10.64}$$

where $R_{eq} = 4 + 6 = 10 \text{ k}\Omega$ and the capacitor is now in discharging phase.

The solution of Eq.(10.64) can be found using the initial and final voltage of the capacitor (initial voltage $v_c(t = \tau = 0.1) = 25.28 \text{ V}$, $v_c(t - \tau = \infty) = 0 \text{ V}$) and it is given by

$$v_c(t) = v_c(\tau = 0.1) \times e^{-\frac{1}{R_{eq}C}(t-\tau)} = 25.28 \times e^{-10(t-\tau)} \tag{10.65}$$

Discharging current expression is given by (note, current direction is just opposite to the assigned direction and it is taken into account with a -ve sign)

$$i_c(t) = -\frac{v_c(t)}{R_{eq}} = -\frac{25.28 \times e^{-10(t-\tau)}}{10} = -2.528 \times e^{-10(t-\tau)} \text{ (in mA)} \tag{10.66}$$

(Note, the above two expressions are valid only for $t \geq \tau$)

The circuit responses for charging and discharging phases in (i) and (ii) are shown in fig.10.25 (d).

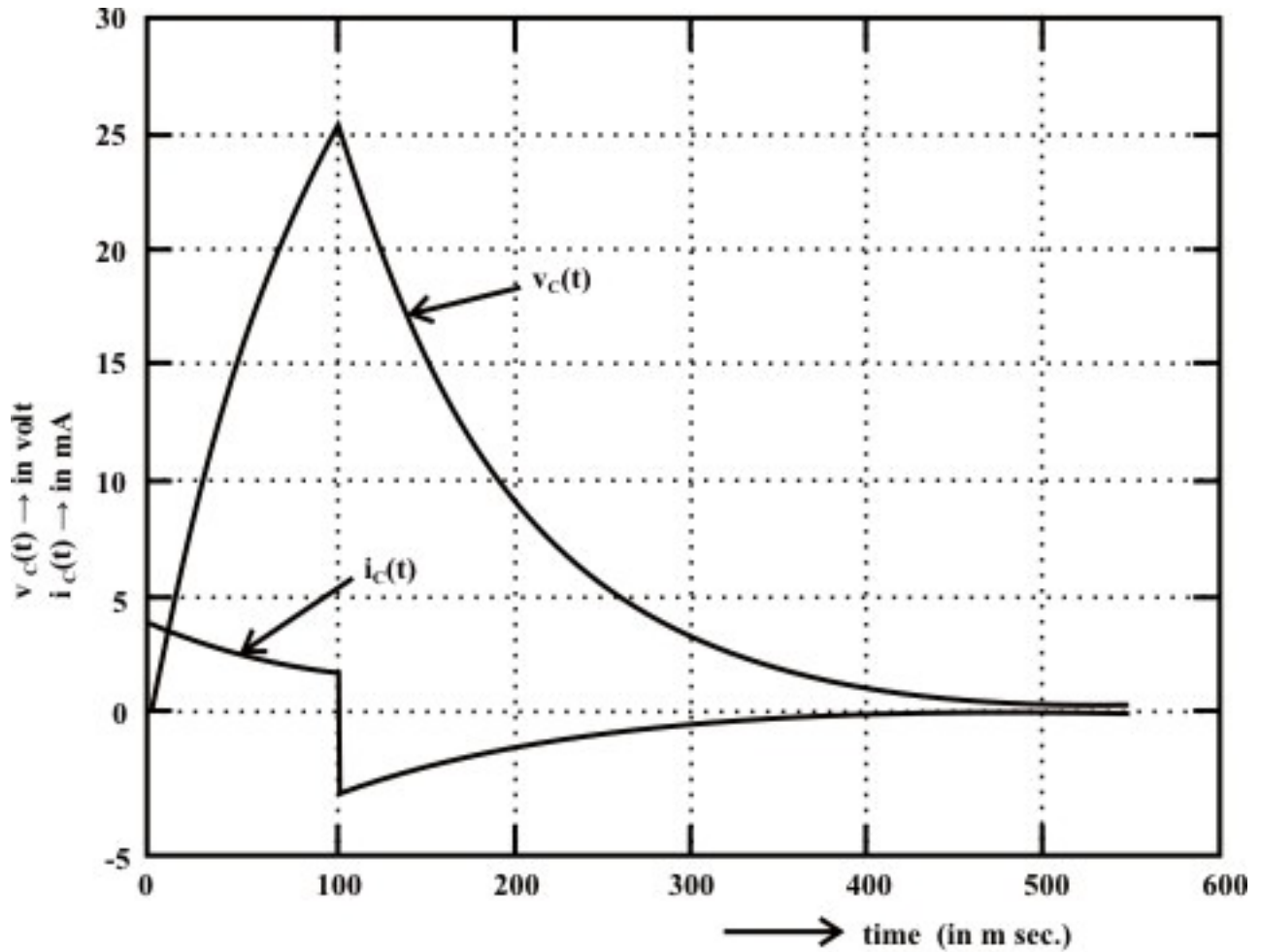


Fig. 10.25(d): Complete response of the circuit indicating charging and discharging phases.

Remark-6 Note that the current through the capacitor (see fig. 10.25(d)) can change instantaneously like the voltage across the inductor.

Appendix-A

L.10.A Solution of n^{th} order linear time invariant differential equation excited by forcing function.

Let us consider a linear time invariant circuit having several energy source elements is described by the following dynamic equation.

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + a_{n-2} \frac{d^{n-2} x}{dt^{n-2}} + \dots + a_1 \frac{dx}{dt} + a_0 x = f(t) \quad (10.A1)$$

where $a_1, a_2, a_3, \dots, a_{n-1}, a_n$ are constant coefficients associated in the differential equation and they are dependent on circuit parameters (like, $R, L,$ and C for electric circuit) but independent of time, $f(t)$ is the forcing or driving function and $x(t)$ is the solution of differential equation or response of the system. We shall discuss the

solution of differential equation restricted to second order differential, say $n=2$ in equation (10.A1).

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t) \quad (10.A2)$$

The solution of this differential equation provides the response of circuit and it is given by

$$x(t) = x_n(t) + x_f(t) \quad (10.A3)$$

where $x_n(t)$ is the natural response of circuit, obtained by setting $f(t) = 0$, and $x_f(t)$ is the forced response that satisfies the original differential equation (10.A2).

By setting $f(t) = 0$ in equation (10.A2), as given in equation (10.A4), the force free equation is obtained.

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \quad (\text{Homogeneous equation}) \quad (10.A4)$$

The solution of such differential equation (or homogeneous equation) is known as natural solution or complementary solution or transient solution and it is denoted by $x_n(t)$. To get the natural solution $x_n(t)$ of equation (10.A4) the following steps are considered.

Let us use the following operators

$$\frac{d}{dt} = \alpha; \quad \frac{d^2}{dt^2} = \frac{d}{dt} \left(\frac{d}{dt} \right) = \alpha^2$$

in equation (10.A4) and results an equation given by

$$(a_2 \alpha^2 + a_1 \alpha + a_0) x = 0$$

Since $x \neq 0$, the above equation can be written as

$$(a_2 \alpha^2 + a_1 \alpha + a_0) = 0 \quad (10.A5)$$

which is known as characteristic equation for a circuit whose force free equation is Eq.(10.A4). The natural or transient solution of Eq.(10.A4) is expressed by the exponential terms as given below.

$$x_n(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \quad (10.A6)$$

where α_1 and α_2 are the roots of characteristic equation (10.A5). The roots of second order characteristic equation with real coefficients is either real or complex occur in conjugate pairs. The constants A_1 and A_2 are evaluated from initial or boundary conditions of circuit. The principles of continuity of inductance current and capacitance voltage are used to establish the required boundary conditions.

If $x_n(t)$ is the natural or transient solution of unforced (or homogeneous) equation differential, it must satisfy its own differential equation

$$a_2 \frac{d^2 x_n}{dt^2} + a_1 \frac{dx_n}{dt} + a_0 x_n = 0 \quad (10.A7)$$

Further, if $x(t) = x_n(t) + x_f(t)$ is the complete solution of given differential Eq.(10.A2), it must satisfy its own equation

$$a_2 \frac{d^2(x_n + x_f)}{dt^2} + a_1 \frac{d(x_n + x_f)}{dt} + a_0(x_n + x_f) = f(t) \quad (10.A8)$$

Using the equation (10.A7) in Eq.(10.A8), we get

$$a_2 \frac{d^2 x_f}{dt^2} + a_1 \frac{dx_f}{dt} + a_0 x_f = f(t) \quad (10.A9)$$

The above equation implies that $x_f(t)$ is the forced solution or steady state solution of second order differential equation (10.A2). Steady state solution of some common forcing functions is listed in Table (assume $a_2 > 0, a_1 > 0$ and $a_0 > 0$).

Table: Steady state solution $x_f(t)$ for any order differential equation excited by some common forcing function.

Type of forcing function $f(t)$ (input)	Steady state solution $x_f(t)$ (output)
• $f(t) = K$ (constant)	• $x_f(t) = A$ (constant)
• $f(t) = Kt$	• $x_f(t) = At + B$
• $f(t) = Kt^2$	• $x_f(t) = At^2 + Bt + C$
• $f(t) = Ke^{at}$	• $x_f(t) = Ae^{at}$
• $f(t) = \sin bt$	• $i_f(t) = A \sin bt + B \cos bt$
• $f(t) = \cos bt$	• $i_f(t) = A \sin bt + B \cos bt$
• $f(t) = e^{at} \sin bt$	• $i_f(t) = e^{at} (A \sin bt + B \cos bt)$
• $f(t) = e^{at} \cos bt$	• $i_f(t) = e^{at} (A \sin bt + B \cos bt)$

Coefficients involve in the steady state solution can be found out by using the boundary conditions of the circuit.

Remark-7

(i) Eq. (10.A2) is the differential equation description of a linear circuit, superposition may be used to find the complete solution of a forcing function which is sum of natural and steady state responses. (ii) Eq.(10.A6) is the natural solution of force-free linear differential equation. Note that the constants α_1 and α_2 are the roots of the characteristic equation (10.A5) and they are entirely depending on the circuit parameters. The roots of the characteristic equation may be classified as

Case-1: Real or Complex but distinct

The natural solution of homogeneous equation (10.A4) is given as

$$x_n(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

Case-2: Roots are repeated (i.e. $\alpha_1 = \alpha_2 = \alpha$ or multiplicity of roots of order 2)

The natural solution of homogeneous equation (10.A4) is given as

$$x_n(t) = \beta_0 e^{\alpha t} + \beta_1 t$$

Using initial and final conditions of the circuit, β_0 and β_1 constants are computed.

More discussions on these issues can be seen in Lesson-11.

L.10.6 Test your understanding

(Marks: 70)

- T.10.1 Inductor tends to block ----- current but pass ----- current.
- T.10.2 The basic fundamental principle that explains the action of an inductor is known as ----- law.
- T.10.3 Exponential waveforms start ----- and finish -----.
- T.10.4 A transient approximately always has a duration of ----- time constants.
- T.10.5 After the first time constant, a transient goes through ----- % of its steady state value.
- T.10.6 ----- through inductor cannot change ----- but ----- across the inductor can ----- instantaneously at the switching phase.
- T.10.7 A simple series $R-L$ circuit is excited with a constant voltage source, the speed of response depends on ----- and ----- of the circuit.
- T.10.8 The energy stored in an inductor in the form of -----.

- T.10.9 In a first order circuit if the resistor value is doubled, the time constant is halved for an ----- circuit.
- T.10.10 An inductor acts as ----- for a ----- current through it.
- T.10.11 Once a capacitor has been charged up, it is able to act like a -----.
- T.10.12 If the spacing between the plates is doubled, the capacitance value is -----.
- T.10.13 After a capacitor is fully charged in a dc circuit, it ----- dc current.
- T.10.14 The time rate of change of capacitor voltage is represented by the ----- tangent line to the $v_c(t)$ -versus- t curve.
- T.10.15 Immediately after a switch has been thrown, a capacitor's ----- must maintain the same value that excited just before the switching instant.
- T.10.16 At the instant of switching, current through the capacitor ----- instantaneously.

- T.10.17 At steady state condition in a dc circuit, the capacitor acts as an ----- circuit.
- T.10.18 A first order circuit with a single resistor, if the resistor is doubled in value, the time constant is also ----- for an $R-C$ circuit.
- T.10.19 Time constant of a first order system is the measure of ----- response of the circuit.
- T.10.20 The energy stored in a capacitor in the form of ----- . [1×20]
- T.10.21 For the circuit of fig.10.26, find (i) $i_1(0^-)$, $i_L(0^-)$ (ii) $i_1(0^+)$, $i_L(0^+)$ (iii) $i_1(t = \infty)$, $i_L(t = \infty)$ (iv) $v_{ab}(0^+)$, $v_{ab}(t = \infty)$.

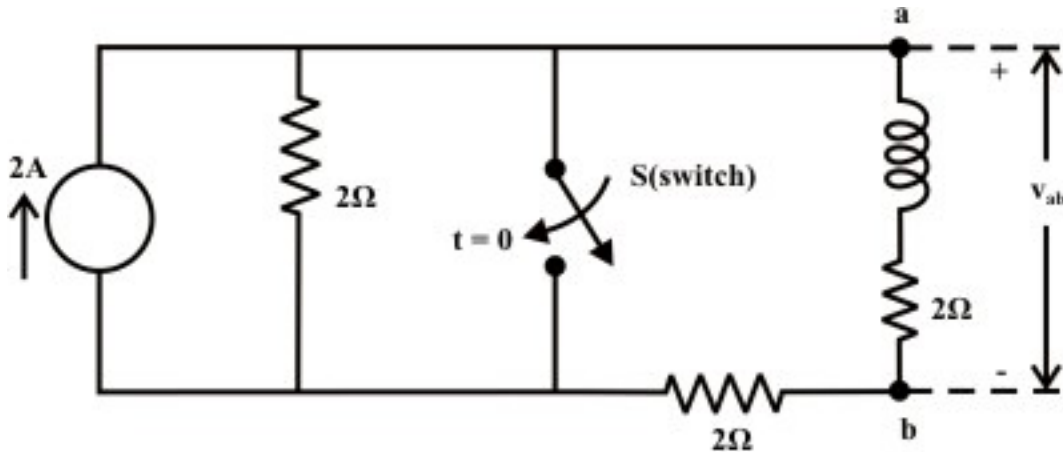


Fig. 10.26

(Ans. (i) 0, 0.666 A (ii) 1.333 A, 0.666 A (iii) 2 A, 0 A (iv) -1.332 V, 0 V) [8]

T.10.22 For the circuit shown in Fig.10.12, the switch 'S' has been opened for a long time and then closes at $t=0$.

Find,

- (i) $v_{ab}(0^-)$ (ii) $i_x(0^-)$ (iii) $i_x(0^+)$ (iv) $v_{ab}(0^+)$ (v) $i_x(t = \infty)$ (vi) $v_{ab}(t = \infty)$ (vii) $i_x(t)$ for $t > 0$

(Ans. [10]

T.10.23 In the circuit shown in fig.10.27, the switch was initially open and no current was flowing in inductor (L). The switch was closed at $t = 0$ and than re opened at $t = 2\tau$

sec. At $t = 0$, $\frac{d_i(t)}{dt}$ was $50 A/s$.

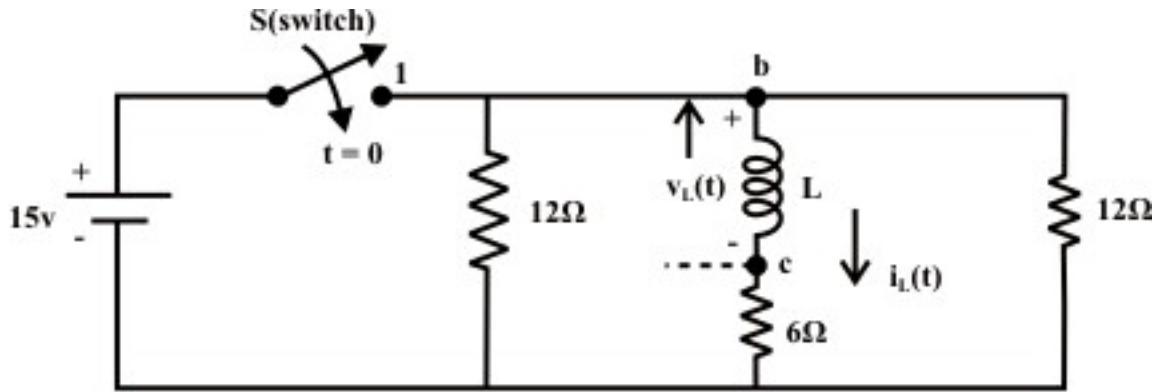


Fig. 10.27

Find,

- (i) The value of L
- (ii) Find the current $i_L(t)$ and voltage $v_{bc}(t)$ expressions for $t \geq 0$. Assume, no current was flowing through the inductor at $t = 0$ (i.e., $i_L(0) = 0$).
- (iii) Find the mathematical expressions for $i_L(t)$ and $v_{bc}(t)$ if the switch 'S' is reopened at $t = 2\tau$ (sec.).
- (iv) Plot the waveforms obtained in parts (ii) to (iii) on the same time axis (time \rightarrow in ms.) for the current $i_L(t)$ and the voltage $v_{bc}(t)$ considering the indicated current directions and identified polarity of voltage across the $b-c$ terminals.

(Ans. (i) $0.3H$ (ii) $i_L(t) = 1.25 \times (1 - e^{-40t})$ amp., $v_{bc}(t) = 15 \times e^{-40t}$

(iii) $i_L(t) = 1.081 \times e^{-40(t-\tau)}$, $v_{bc}(t) = 12.96 \times e^{-40(t-2\tau)}$, for $t \geq 2\tau$.) [10]

T.10.24 At steady state condition, find the values of $I_1, I_2, I_3, I_4, I_5, V_1$ and V_2 for the circuit shown in fig.10.28.

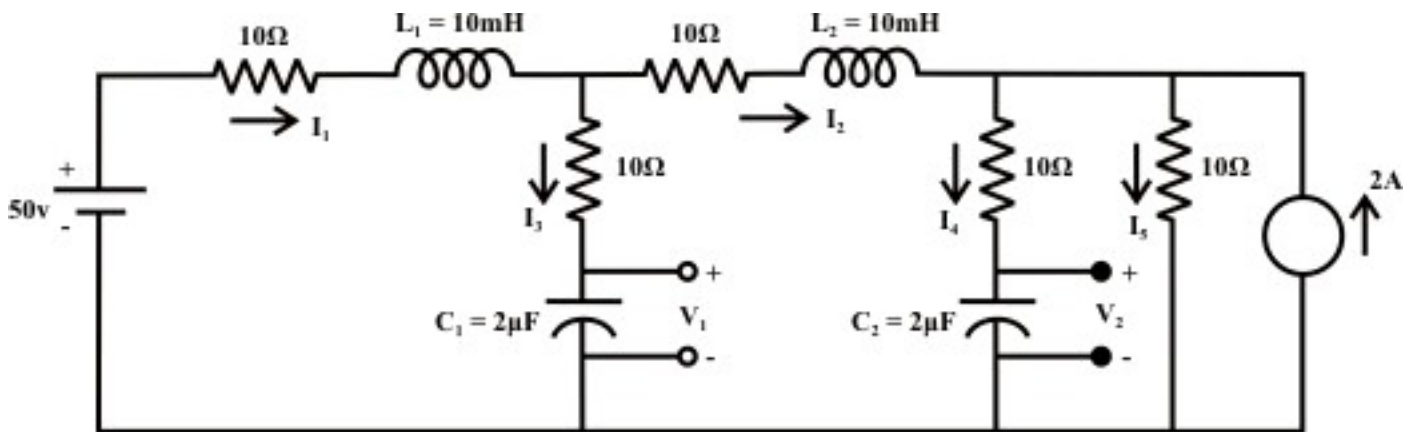


Fig. 10.28

(Ans. $I_1 = I_2 = I_5 = 1A, I_4 = I_3 = 0, V_1 = 40V$ and $V_2 = 30V$)

[6]

T.10.25 Switch 'S' shown in fig.10.29 is kept in position '1' for a long time.

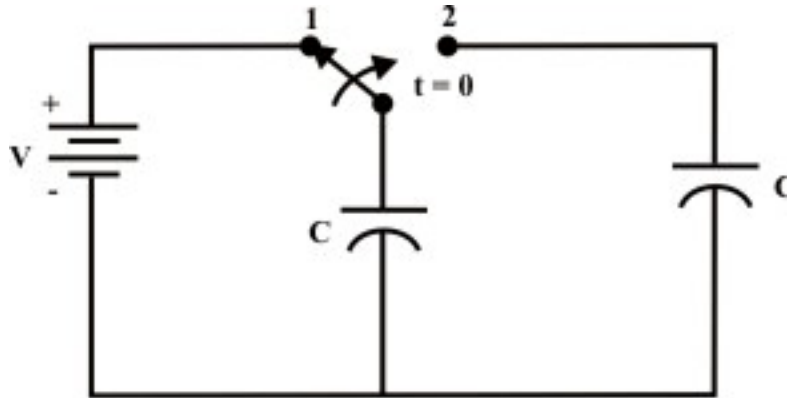


Fig. 10.29

When the switch is thrown in position '2', find at steady state condition

(i) the voltage across the each capacitor (ii) the charge across the each capacitor (iii) the energy stored by the each capacitor

(Ans. (i) (i) $\frac{V}{2}$ (ii) $C \frac{V}{2}$ (iii) $C \frac{V^2}{8}$) [6]

T.10.26 For the circuit shown in fig.10.30, Switch 'S' is kept in position '1' for a long time and then it is thrown in position '2' at time $t=0$. Find (a) the current expression $i(t)$ for $t \geq 0$ (b) calculate the time constants of the circuit before and after the switching phases.

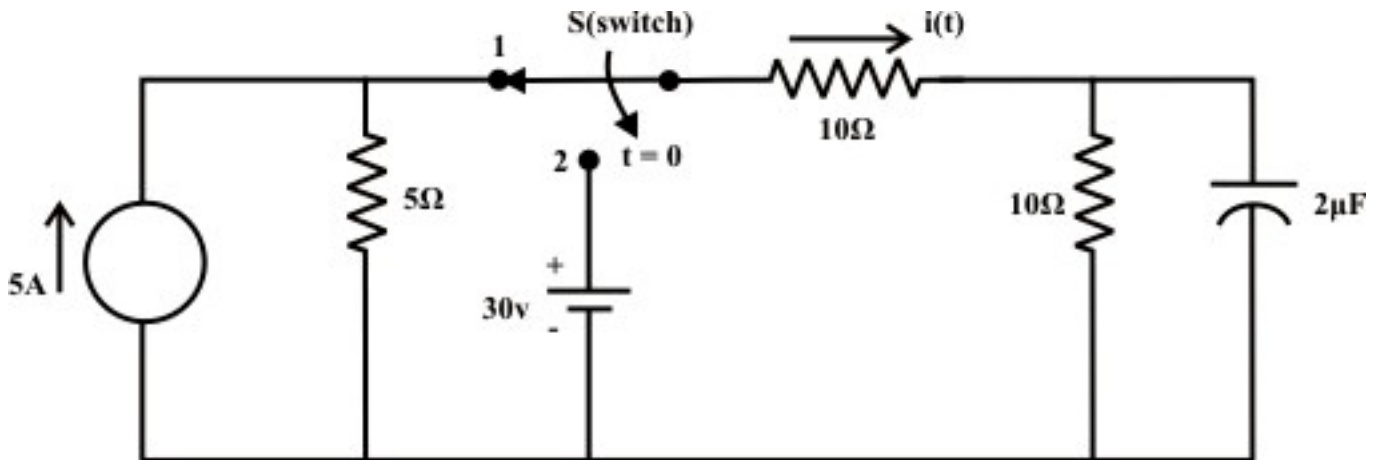


Fig. 10.30

(Ans. (a) $i(t) = 1.5 + 0.5e^{-10^5 t}$ (b) $12 \mu s$ (before the switch is opened), (b) $10 \mu s$ (after the switch is opened, i.e., when the switch is in position '2')) [10]